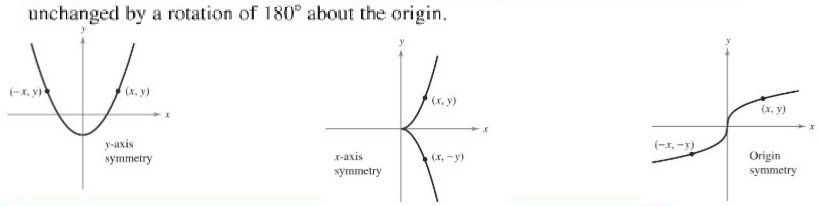
P.1 Graphs and Models

- Sketch the graph of an equation.
- Find the intercepts of a graph.
- Test a graph for symmetry with respect to an axis and the origin.
- Find the points of intersection of two graphs.
- Fit a mathematical model to a real-life data set.

P.1 Graphs and Models

1. A graph is **symmetric with respect to the *y*-axis** if, whenever (x, y) is a point on the graph, then $(-x, y)$ is also a point on the graph. This means that the portion of the graph to the left of the y-axis is a mirror image of the portion to the right of the y-axis.
2. A graph is **symmetric with respect to the *x*-axis** if, whenever (x, y) is a point on the graph, then $(x, -y)$ is also a point on the graph. This means that the portion of the graph below the x-axis is a mirror image of the portion above the x-axis.
3. A graph is **symmetric with respect to the origin** if, whenever (x, y) is a point on the graph, then $(-x, -y)$ is also a point on the graph. This means that the graph is unchanged by a rotation of 180° about the origin.



Tests for Symmetry

1. The graph of an equation in x and y is symmetric with respect to the y-axis when replacing x by $-x$ yields an equivalent equation.
2. The graph of an equation in x and y is symmetric with respect to the x-axis when replacing y by $-y$ yields an equivalent equation.
3. The graph of an equation in x and y is symmetric with respect to the origin when replacing x by $-x$ and y by $-y$ yields an equivalent equation.

P.1 Graphs and Models

- Sketch the graph of an equation.
- Find the intercepts of a graph.
- Test a graph for symmetry with respect to an axis and the origin.

Example 1: Graph the following, then test for symmetry.

a. $y = x^3 - 4x$

b. $y = x^2 - 4$

x	y

x	y

P.1 Graphs and Models

- Find the points of intersection of two graphs.

Example 2: Find all points of intersection of the graphs of $y + x^2 = 9$ and $y - x = 3$.

P.1 Graphs and Models

- Fit a mathematical model to a real-life data set.

Example 3: Simple harmonic motion can be modeled by a sine or cosine function. Write the equation for the simple harmonic motion that satisfies the specified conditions.

Displacement (t=0)	0 cm
Amplitude	4 cm
Period	6 sec

P.2 Linear Models and Rates of Change

- Find the slope of a line passing through two points.
- Write the equation of a line with a given point and slope.
- Interpret slope as a ratio or as a rate in a real-life application.
- Sketch the graph of a linear equation in slope-intercept form.
- Write equations of lines that are parallel or perpendicular to a given line.

P.2 Linear Models and Rates of Change

- Find the slope of a line passing through two points.
- Interpret slope as a ratio or as a rate in a real-life application.

Definition of the Slope of a Line

The **slope** m of the nonvertical line passing through (x_1, y_1) and (x_2, y_2) is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}, \quad x_1 \neq x_2.$$

Slope is not defined for vertical lines.

Ratios and Rates of Change

The slope of a line can be interpreted as either a *ratio* or a *rate*. If the x- and y-axes have the same unit of measure, then the slope has no units and is a **ratio.** If the x- and y-axes have different units of measure, then the slope is a rate or **rate of change.** In your study of calculus, you will encounter applications involving both interpretations of slope.

The rate of change found in Example 3 is an **average rate of change.** An average rate of change is always calculated over an interval. In this case, the interval is [2000, 2012]. In Chapter 2, you will study *instantaneous rate of change.*

P.2 Linear Models and Rates of Change

- Find the slope of a line passing through two points.
- Interpret slope as a ratio or as a rate in a real-life application.

Example 1: Answer the following slope questions involving ratios and/or rates of change.

a. When driving down a mountainside you notice warning signs indicating that the road has a grade of 12%. This means that the slope of the road is $-\frac{12}{100}$. Approximate the amount of horizontal change in your position if you have descended 2000 feet vertically.

b. The nets sales for a car manufacturer wee \$14.61 billion in 2011 and \$16.95 billion in 2013. Find the average rate of change of the net sales over this 2-year period. What will the net sales for the car manufacturer be in 2015?

P.2 Linear Models and Rates of Change

- Write the equation of a line with a given point and slope.
- Sketch the graph of a linear equation in slope-intercept form.
- Write equations of lines that are parallel or perpendicular to a given line.

Point-Slope Form of the Equation of a Line

The **point-slope form** of the equation of the line that passes through the point (x_1, y_1) and has a slope of m is

$$y - y_1 = m(x - x_1).$$

Remember that only nonvertical lines have a slope. Consequently, vertical lines cannot be written in point-slope form.

The Slope-Intercept Form of the Equation of a Line

The graph of the linear equation

$$y = mx + b$$ Slope-intercept form

is a line whose slope is m and whose y-intercept is $(0, b)$.

Summary of Equations of Lines

1. General form: $Ax + By + C = 0$
2. Vertical line: $x = a$
3. Horizontal line: $y = b$
4. Slope-intercept form: $y = mx + b$
5. Point-slope form: $y - y_1 = m(x - x_1)$

P.2 Linear Models and Rates of Change

- Write the equation of a line with a given point and slope.
- Sketch the graph of a linear equation in slope-intercept form.
- Write equations of lines that are parallel or perpendicular to a given line.

Parallel and Perpendicular Lines

1. Two distinct nonvertical lines are **parallel** if and only if their slopes are equal—that is, if and only if

   $m_1 = m_2$. Parallel ⟺ Slopes are equal.

2. Two nonvertical lines are **perpendicular** if and only if their slopes are negative reciprocals of each other—that is, if and only if

   $m_1 = -\frac{1}{m_2}$. Perpendicular ⟺ Slopes are negative reciprocals.

Remark

The term *normal* may be used to refer to lines that are perpendicular.

P.2 Linear Models and Rates of Change

- Write the equation of a line with a given point and slope.
- Sketch the graph of a linear equation in slope-intercept form.
- Write equations of lines that are parallel or perpendicular to a given line.

Example 2: Given a line that has a slope of 2 and passes though the point (3,-2), find the following:

a. Equation of the line in Slope-intercept form.
b. Sketch a graph of the equation of the line.
c. Equation of a line parallel to it and passing though the point (-1,2).
d. Equation of a line perpendicular to it and passing through the point (3,0).

P.3 Functions and Their Graphs

- Use function notation to represent and evaluate a function.
- Find the domain and range of a function.
- Sketch the graph of a function.
- Identify different types of transformations of functions.
- Classify functions and recognize combinations of functions.

P.3 Functions and Their Graphs

- Use function notation to represent and evaluate a function.
- Find the domain and range of a function.

Definition of a Real-Valued Function of a Real Variable

Let X and Y be sets of real numbers. A **real-valued function f of a real variable x** from X to Y is a correspondence that assigns to each number x in X exactly one number y in Y.

The **domain** of f is the set X. The number y is the **image** of x under f and is denoted by $f(x)$, which is called the **value of f at x.** The **range** of f is a subset of Y and consists of all images of numbers in X. (See Figure P.24.)

The Domain and Range of a Function

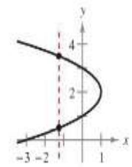
The domain of a function can be described *explicitly*, or it may be described *implicitly* by an equation used to define the function. The **implied domain** is the set of all real numbers for which the equation is defined, whereas an explicitly defined domain is one that is given along with the function. Below are two examples.

$f(x) = \dfrac{1}{x^2 - 4}, \quad 4 \le x \le 5$ — Explicitly defined domain $\{x: 4 \le x \le 5\}$

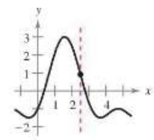
$g(x) = \dfrac{1}{x^2 - 4}$ — Implied domain $\{x: x \ne \pm 2\}$

A function from X to Y is **one-to-one** when to each y-value in the range there corresponds exactly one x-value in the domain. For instance, the function in Example 2(a) is one-to-one, whereas the functions in Examples 2(b) and 2(c) are not one-to-one. A function from X to Y is **onto** when its range consists of all of Y.

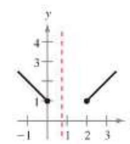
Vertical Line Test,



(a) Not a function of x



(b) A function of x



(c) A function of x

P.3 Functions and Their Graphs

- Use function notation to represent and evaluate a function.
- Find the domain and range of a function.

Example 1: For the function f defined by $f(x) = 8 - 2x^2$, evaluate each expression.

a. $f(2)$

b. $f(x-3)$

c. $\dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

Example 2: Find the domain and range of each function.

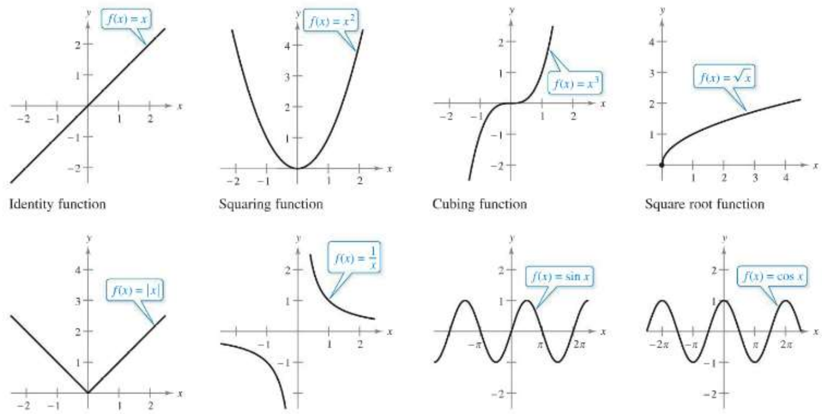
a. $f(x) = \sqrt{49 - x^2}$

b. $g(x) = \sec x$

c. $h(x) = \begin{cases} 2 - x, x \le 2 \\ \sqrt{x-2}, x > 2 \end{cases}$

P.3 Functions and Their Graphs

▶ Sketch the graph of a function.



The graphs of eight basic functions

P.3 Functions and Their Graphs

▶ Sketch the graph of a function.

elementary functions.

1. Algebraic functions (polynomial, radical, rational)
2. Trigonometric functions (sine, cosine, tangent, and so on)
3. Exponential and logarithmic functions

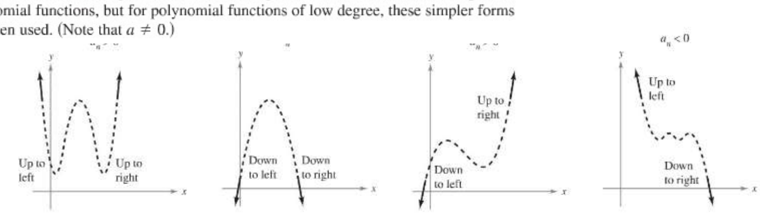
The most common type of algebraic function is a polynomial function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2x^2 + a_1x + a_0$$

where n is a nonnegative integer. The numbers a_i are **coefficients,** with a_n the **leading coefficient** and a_0 the **constant term** of the polynomial function. If $a_n \neq 0$, then n is the **degree** of the polynomial function. The zero polynomial $f(x) = 0$ is not assigned a degree. It is common practice to use subscript notation for coefficients of general polynomial functions, but for polynomial functions of low degree, these simpler forms are often used. (Note that $a \neq 0$.)

Remark

Polynomial functions of the form $f(x) = x^n$ are often referred to as *power functions.*



Graphs of polynomial functions of even degree

Graphs of polynomial functions of odd degree

The Leading Coefficient Test for polynomial functions

P.3 Functions and Their Graphs

▶ Identify different types of transformations of functions.

Basic Types of Transformations ($c > 0$)

Original graph:	$y = f(x)$
Horizontal shift c units to the **right:**	$y = f(x - c)$
Horizontal shift c units to the **left:**	$y = f(x + c)$
Vertical shift c units **downward:**	$y = f(x) - c$
Vertical shift c units **upward:**	$y = f(x) + c$
Reflection (about the x-axis):	$y = -f(x)$
Reflection (about the y-axis):	$y = f(-x)$
Reflection (about the origin):	$y = -f(-x)$

P.3 Functions and Their Graphs

▶ Classify functions and recognize combinations of functions.

Just as a rational number can be written as the quotient of two integers, a **rational function** can be written as the quotient of two polynomials. Specifically, a function f is rational when it has the form

$$f(x) = \frac{p(x)}{q(x)}, \quad q(x) \neq 0$$

where $p(x)$ and $q(x)$ are polynomials.

Polynomial functions and rational functions are examples of **algebraic functions.** An algebraic function of x is one that can be expressed as a finite number of sums, differences, multiples, quotients, and radicals involving x^n. For example,

$$f(x) = \sqrt{x + 1}$$

is algebraic. Functions that are not algebraic are **transcendental.** For instance, the trigonometric functions are transcendental.

P.3 Functions and Their Graphs

▶ Classify functions and recognize combinations of functions.

There are various ways to combine two functions to create new functions. For example, given $f(x) = 2x - 3$ and $g(x) = x^2 + 1$, you can form the functions shown.

$(f + g)(x) = f(x) + g(x) = (2x - 3) + (x^2 + 1)$ Sum

$(f - g)(x) = f(x) - g(x) = (2x - 3) - (x^2 + 1)$ Difference

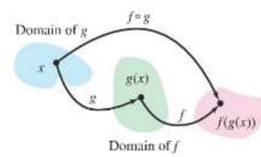
$(fg)(x) = f(x)g(x) = (2x - 3)(x^2 + 1)$ Product

$(f/g)(x) = \dfrac{f(x)}{g(x)} = \dfrac{2x - 3}{x^2 + 1}$ Quotient

You can combine two functions in yet another way, called **composition.** The resulting function is called a **composite function.**

Definition of Composite Function

Let f and g be functions. The function $(f \circ g)(x) = f(g(x))$ is the **composite** of f with g. The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. (See Figure P.30.)



The domain of the composite function $f \circ g$

Figure P.30

The composite of f with g is generally not the same as the composite of g with f. This is shown in the next example.

Test for Even and Odd Functions

The function $y = f(x)$ is **even** when $f(-x) = f(x)$.

The function $y = f(x)$ is **odd** when $f(-x) = -f(x)$.

P.3 Functions and Their Graphs

▶ Classify functions and recognize combinations of functions.

Example 3: For $f(x) = 3x + 4$ and $g(x) = 2x^2 + 2$, find each composite function.

a. $f \circ g$

b. $g \circ f$

Example 4: Determine whether the following functions are even, odd, or neither. Then find the zeros of the functions.

a. $f(x) = 3x^2 + 4x - 7$

b. $g(x) = 1 - \cos x$

P.4 Inverse Functions

- Verify that one function is the inverse function of another function.
- Determine whether a function has an inverse function.
- Develop properties of the six inverse trigonometric functions.

Definition of Inverse Function

A function g is the **inverse function** of the function f when

$f(g(x)) = x$ for each x in the domain of g

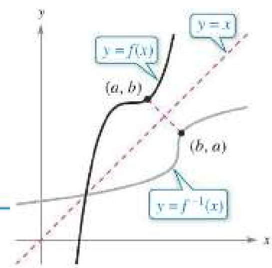
and

$g(f(x)) = x$ for each x in the domain of f.

The function g is denoted by f^{-1} (read "f inverse").

Reflective Property of Inverse Functions

The graph of f contains the point (a, b) if and only if the graph of f^{-1} contains the point (b, a).



The graph of f^{-1} is a reflection of the graph of f in the line $y = x$.

The Existence of an Inverse Function

A function has an inverse function if and only if it is one-to-one.

P.4 Inverse Functions

- Verify that one function is the inverse function of another function.
- Determine whether a function has an inverse function.
- Develop properties of the six inverse trigonometric functions.

Example 1: Show that the functions $f(x) = x^2 + 1, x \geq 0$, and $g(x) = \sqrt{x-1}$ are inverse functions of each other.

Example 2: Use the graph of the function $f(x) = \frac{3-x}{2}$ and the Horizontal Line Test to determine whether the function has an inverse.

P.4 Inverse Functions

Guidelines for Finding an Inverse of a Function

1. Determine whether the function given by $y = f(x)$ has an inverse function.
2. Solve for x as a function of y: $x = g(y) = f^{-1}(y)$.
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.
4. Define the domain of f^{-1} as the range of f.
5. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

Example 3: Find the inverse function of $f(x) = \sqrt[3]{7+x}$.

P.4 Inverse Functions

- Develop properties of the six inverse trigonometric functions.

Definition of Inverse Trigonometric Functions

Function	*Domain*	*Range*
$y = \arcsin x$ iff $\sin y = x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \arccos x$ iff $\cos y = x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \arctan x$ iff $\tan y = x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \operatorname{arccot} x$ iff $\cot y = x$	$-\infty < x < \infty$	$0 < y < \pi$
$y = \operatorname{arcsec} x$ iff $\sec y = x$	$\lvert x \rvert \ge 1$	$0 \le y \le \pi, \quad y \ne \frac{\pi}{2}$
$y = \operatorname{arccsc} x$ iff $\csc y = x$	$\lvert x \rvert \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}, \quad y \ne 0$

P.4 Inverse Functions

▶ Develop properties of the six inverse trigonometric functions.

The graphs of the six inverse trigonometric functions are shown in Figure P.39.

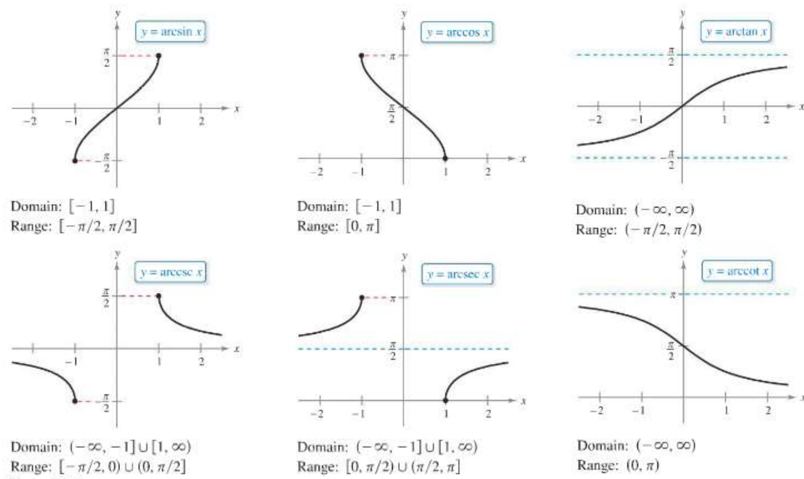


Figure P.39

P.4 Inverse Functions

▶ Develop properties of the six inverse trigonometric functions.

Properties of Inverse Trigonometric Functions

1. If $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$, then

$$\sin(\arcsin x) = x \quad \text{and} \quad \arcsin(\sin y) = y.$$

2. If $-\pi/2 < y < \pi/2$, then

$$\tan(\arctan x) = x \quad \text{and} \quad \arctan(\tan y) = y.$$

3. If $|x| \ge 1$ and $0 \le y < \pi/2$ or $\pi/2 < y \le \pi$, then

$$\sec(\operatorname{arcsec} x) = x \quad \text{and} \quad \operatorname{arcsec}(\sec y) = y.$$

Similar properties hold for the other inverse trigonometric functions.

Example 3: Evaluate each expression:

a. $\arccos \frac{\sqrt{3}}{2}$ b. $\cos^{-1} 0.5$ c. $\arctan 1$ d. $\sin^{-1} -0.919$

P.4 Inverse Functions

- Develop properties of the six inverse trigonometric functions.

Example 4: Solve the following.

a. $\arcsin\left(\frac{3x-}{4}\right) = \frac{\pi}{6}\ for\ x$

b. Given $y = \arccos 2x$, where $0 < y < \frac{\pi}{2}$, find $\tan y$.

P.5 Exponential and Logarithmic Functions

- Develop and use properties of exponential functions.
- Understand the definition of the number *e*.
- Understand the definition of the natural logarithmic function, and develop and use properties of the natural logarithmic function.

Properties of Exponents

Let a and b be positive real numbers, and let x and y be any real numbers.

1. $a^0 = 1$ **2.** $a^x a^y = a^{x+y}$ **3.** $(a^x)^y = a^{xy}$ **4.** $(ab)^x = a^x b^x$

5. $\frac{a^x}{a^y} = a^{x-y}$ **6.** $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$ **7.** $a^{-x} = \frac{1}{a^x}$

Properties of Exponential Functions

Let a be a real number that is greater than 1.

1. The domain of $f(x) = a^x$ and $g(x) = a^{-x}$ is $(-\infty, \infty)$.
2. The range of $f(x) = a^x$ and $g(x) = a^{-x}$ is $(0, \infty)$.
3. The y-intercept of $f(x) = a^x$ and $g(x) = a^{-x}$ is $(0, 1)$.
4. The functions $f(x) = a^x$ and $g(x) = a^{-x}$ are one-to-one.

P.5 Exponential and Logarithmic Functions

- Develop and use properties of exponential functions.
- Understand the definition of the number *e*.
- Understand the definition of the natural logarithmic function, and develop and use properties of the natural logarithmic function.

Connecting Notations
The notation $\ln x$ is read as "el en of x" or "the natural log of x."

Definition of the Natural Logarithmic Function

Let x be a positive real number. The **natural logarithmic function,** denoted by $\ln x$, is defined as

$\ln x = b$ if and only if $e^b = x$.

Properties of the Natural Logarithmic Function

1. The domain of $g(x) = \ln x$ is $(0, \infty)$.
2. The range of $g(x) = \ln x$ is $(-\infty, \infty)$.
3. The x-intercept of $g(x) = \ln x$ is $(1, 0)$.
4. The function $g(x) = \ln x$ is one-to-one.

Properties of Logarithms

Let x, y, and z be real numbers such that $x > 0$ and $y > 0$.

1. $\ln xy = \ln x + \ln y$
2. $\ln \frac{x}{y} = \ln x - \ln y$
3. $\ln x^z = z \ln x$

P.5 Exponential and Logarithmic Functions

- Understand the definition of the natural logarithmic function, and develop and use properties of the natural logarithmic function.

Example 1: Expand the following Logarithmic Expressions.

a. $\ln 3x^2 y$

b. $\ln \frac{\sqrt{4x+1}}{8}$

Example 2: Solve for x.

a. $e^{3x-7} = 2$

b. $\ln(6 - x) = 3$