2.4 The Chain Rule

- Find the derivative of a composite function using the Chain Rule.
- Find the derivative of a function using the General Power Rule.
- Simplify the derivative of a function using algebra.
- Find the derivative of a transcendental function using the Chain Rule.
- Find the derivative of a function involving the natural logarithmic function.
- Define and differentiate exponential functions that have bases other than *e*.

THEOREM 2.11 The Chain Rule

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$

Insight

On the AP® Exam, the Chain Rule is just as widely tested conceptually as it is analytically.

2.4 The Chain Rule

THEOREM 2.12 The General Power Rule

If $y = [u(x)]^n$, where u is a differentiable function of x and n is a rational number, then

$$\frac{dy}{dx} = n[u(x)]^{n-1}\frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[u^n] = nu^{n-1}u'.$$

Example 1: For each function $y = f(g(x))$, identify $u = g(x)$ and $y = f(u)$.

a. $y = (x^2 - 1)^3$

b. $y = \sin x^2$

c. $y = \frac{2}{(x+7)^4}$

Example 2: Find $\frac{dy}{dx}$ for $y = g(x)$ and $y = \frac{1}{(3x-5)^2}$.

2.4 The Chain Rule

Example 3: Find the derivative of each function.

a. $f(x) = (5x^2 + 2x)^7$

b. $g(t) = \frac{6}{(3t^2-5)^5}$

Example 4: Find all points on the graph of $f(x) = \sqrt[5]{(2x^2 - 8)^4}$ for which $f'(x) = 0$ and those for which $f'(x)$ does not exist.

2.4 The Chain Rule

Example 5: Find the derivative of each function.

a. $f(x) = 5x\sqrt[3]{(3x + 1)^2}$

b. $f(x) = \frac{x^2}{\sqrt{x^2-3}}$

c. $f(x) = \left(\frac{4x-1}{x^2-2}\right)^2$

2.4 The Chain Rule

Example 6: Find the derivative of each transcendental function.

a. $y = \tan(x^2 + 1)$

b. $y = e^{\frac{x}{4}}$

c. $y = \csc x^2$

d. $y = \csc^2 x$

e. $f(t) = \cos^2 t^3$

2.4 The Chain Rule

THEOREM 2.13 Derivative of the Natural Logarithmic Function

Let u be a differentiable function of x.

1. $\frac{d}{dx}[\ln x] = \frac{1}{x}, \quad x > 0$
2. $\frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx} = \frac{u'}{u}, \quad u > 0$

Insight

The derivative of the natural logarithmic function is frequently tested on the AP® Exam.

Example 7: Find the derivative of each transcendental function.

a. $y = \ln(2x^3)$

b. $y = x^2 \ln x$

c. $y = [\ln(x + 5)]^2$

d. $f(x) = \ln(x - 3)^2$

e. $f(x) = \ln \frac{x^3(x-3)}{\sqrt{x^2-9}}$

2.4 The Chain Rule

THEOREM 2.14 Derivative Involving Absolute Value

If u is a differentiable function of x such that $u \neq 0$, then

$$\frac{d}{dx}[\ln|u|] = \frac{u'}{u}.$$

Definition of Exponential Function to Base a

If a is a positive real number $(a \neq 1)$ and x is any real number, then the **exponential function to the base a** is denoted by a^x and is defined by

$$a^x = e^{(\ln a)x}.$$

If $a = 1$, then $y = 1^x = 1$ is a constant function.

Definition of Logarithmic Function to Base a

If a is a positive real number $(a \neq 1)$ and x is any positive real number, then the **logarithmic function to the base a** is denoted by $\log_a x$ and is defined as

$$\log_a x = \frac{1}{\ln a} \ln x.$$

Insight

Differentiation with bases other than e are rarely tested on the AP® Exam.

2.4 The Chain Rule

THEOREM 2.15 Derivatives for Bases Other than e

Let a be a positive real number $(a \neq 1)$ and let u be a differentiable function of x.

1. $\frac{d}{dx}[a^x] = (\ln a)a^x$
2. $\frac{d}{dx}[a^u] = (\ln a)a^u \frac{du}{dx}$
3. $\frac{d}{dx}[\log_a x] = \frac{1}{(\ln a)x}$
4. $\frac{d}{dx}[\log_a u] = \frac{1}{(\ln a)u} \frac{du}{dx}$

Example 8: Find the derivative of each transcendental function.

a. $y = 7^x$

b. $y = 7^{\frac{x}{2}}$

c. $y = \log_4 \csc x$

d. $y = \log_5 \frac{x+3}{(2x-1)^2}$

2.4 The Chain Rule

Summary of Differentiation Rules

General Differentiation Rules — Let u and v be differentiable functions of x.

Constant Rule: $\frac{d}{dx}[c] = 0$, c is a real number.

(Simple)Power Rule: $\frac{d}{dx}[x^n] = nx^{n-1}$, $\frac{d}{dx}[x] = 1$, n is a rational number.

Constant Multiple Rule: $\frac{d}{dx}[cu] = cu'$, c is a real number.

Sum or Difference Rule: $\frac{d}{dx}[u \pm v] = u' \pm v'$

Product Rule: $\frac{d}{dx}[uv] = uv' + vu'$

Quotient Rule: $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$

Chain Rule: $\frac{d}{dx}[f(u)] = f'(u)u'$

General Power Rule: $\frac{d}{dx}[u^n] = nu^{n-1}u'$, n is a rational number.

Derivatives of Trigonometric Functions

$\frac{d}{dx}[\sin x] = \cos x$ $\quad$ $\frac{d}{dx}[\tan x] = \sec^2 x$

$\frac{d}{dx}[\sec x] = \sec x \tan x$ $\quad$ $\frac{d}{dx}[\cos x] = -\sin x$

$\frac{d}{dx}[\cot x] = -\csc^2 x$ $\quad$ $\frac{d}{dx}[\csc x] = -\csc x \cot x$

Derivatives of Exponential and Logarithmic Functions

$\frac{d}{dx}[e^x] = e^x$ $\quad$ $\frac{d}{dx}[\ln x] = \frac{1}{x}, \quad x > 0$

$\frac{d}{dx}[a^x] = (\ln a)a^x$, a is a positive real number $(a \neq 1)$.

$\frac{d}{dx}[\log_a x] = \frac{1}{(\ln a)x}$, a is a positive real number $(a \neq 1)$.

2.5 Implicit Differentiation

- Distinguish between functions written in implicit form and explicit form.
- Use implicit differentiation to find the derivative of a function.
- Find derivatives of functions using logarithmic differentiation.

Example 1: Evaluate each expression.

a. $\frac{d}{dx}[7x^3]$

b. $\frac{d}{dx}[7y^3]$

c. $\frac{d}{dx}[x^4 - 2y]$

d. $\frac{d}{dx}\left[\frac{x^2}{y^3}\right]$

2.5 Implicit Differentiation

Guidelines for Implicit Differentiation

1. Differentiate both sides of the equation *with respect to x*.
2. Collect all terms involving dy/dx on one side of the equation and move all other terms to the other side of the equation.
3. Factor out dy/dx.
4. Solve for dy/dx.

Example 2: Find $\frac{dy}{dx}$ given that $2y^4 - y^3x^2 + 4x^3 = 5$

2.5 Implicit Differentiation

Example 3: If possible, represent y as a differentiable function of x.

a. $(x-1)^2 + y^2 = 1$ b. $x^2 + (y-5)^2 = 1$ c. $(x-4)^2 + (y-5)^2 = 0$

Example 4: Determine the slope of the tangent line to the graph of $\frac{x^2}{4} + y^2 = 6$ at the point $(2, \sqrt{5})$.

2.5 Implicit Differentiation

Example 5: Determine the slope of the tangent line to the graph of $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 5$ at the point $(8,1)$.

Example 6: Find $\frac{dy}{dx}$ implicitly of the equation $\sin 2y = x$. Then find the largest interval of the form $-a < y < a$ on which y is a differentiable function of x. Write $\frac{dy}{dx}$ as a function of x.

2.5 Implicit Differentiation

Example 7: Given $(\mathrm{x} - 5)^2 + y^2 = 36$, find $\frac{d^2y}{dx^2}$.

Example 8: Find the tangent line to the graph of $(x + 2)^2 + (y - 3)^2 = 37$ at the point $(4,4)$.

2.5 Implicit Differentiation

Example 9: Find the following derivatives.

a. $y = \frac{(3x+5)^2}{(x^2-1)^3}, x \neq -\frac{5}{3}$

b. $y = x^{\frac{x}{4}}, x > 0$

2.6 Derivatives of Inverse Functions

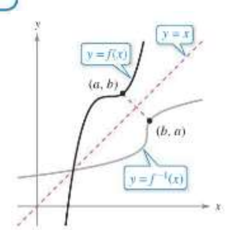
- Find the derivative of an inverse function.
- Differentiate an inverse trigonometric function.

THEOREM 2.16 Continuity and Differentiability of Inverse Functions

Let f be a function whose domain is an interval I. If f has an inverse function, then the following statements are true.

1. If f is continuous on its domain, then f^{-1} is continuous on its domain.
2. If f is differentiable on an interval containing c and $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$.

A proof of this theorem is given in Appendix A.



The graph of f^{-1} is a reflection of the graph of f in the line $y = x$.

2.6 Derivatives of Inverse Functions

THEOREM 2.17 The Derivative of an Inverse Function

Let f be a function that is differentiable on an interval I. If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))}, \quad f'(g(x)) \neq 0.$$

A proof of this theorem is given in Appendix A.

$$\frac{dy}{dx} = \frac{1}{dx/dy}.$$

Example 1: Let $f(x) = 3x^3 - x^2 - 8$.

a. What is the value of $f^{-1}(x)$ when $x = 12$?

a. What is the value of $(f^{-1})'(x)$ when $x = 12$?

2.6 Derivatives of Inverse Functions

Example 2: Let $f(x) = 2x^3$ and let $f^{-1}(x) = \sqrt[3]{\frac{x}{2}}$. Show that the slopes of the graphs of f and f^{-1} are reciprocals at each of the following points.

a. $(1,2)$ and $(2,1)$

b. $(-2,-16)$ and $(-16,-2)$

Example 3: Find the derivative of the inverse cosine function.

2.6 Derivatives of Inverse Functions

THEOREM 2.18 Derivatives of Inverse Trigonometric Functions

Let u be a differentiable function of x.

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2} \qquad \frac{d}{dx}[\text{arccot } u] = \frac{-u'}{1+u^2}$$

$$\frac{d}{dx}[\text{arcsec } u] = \frac{u'}{|u|\sqrt{u^2-1}} \qquad \frac{d}{dx}[\text{arccsc } u] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

Insight

Differentiation of inverse trigonometric functions is sometimes tested on the AP® Exam. The inverse sine and inverse tangent functions are the most likely topics.

Example 3: Find the derivative of each function.

a. $f(x) = \text{arccsc}\, x^2$

b. $f(x) = \arccos 4x$

c. $f(x) = \text{arccot}\, e^{\frac{x}{3}}$

d. $f(x) = \arccos x - x\sqrt{1-x^2}$

2.6 Derivatives of Inverse Functions

Basic Differentiation Rules for Elementary Functions

1. $\frac{d}{dx}[cu] = cu'$	2. $\frac{d}{dx}[u \pm v] = u' \pm v'$	3. $\frac{d}{dx}[uv] = uv' + vu'$
4. $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$	5. $\frac{d}{dx}[c] = 0$	6. $\frac{d}{dx}[u^n] = nu^{n-1}u'$
7. $\frac{d}{dx}[x] = 1$	8. $\frac{d}{dx}[\lvert u\rvert] = \frac{u}{\lvert u\rvert}(u'),\quad u \neq 0$	9. $\frac{d}{dx}[\ln u] = \frac{u'}{u}$
10. $\frac{d}{dx}[e^u] = e^u u'$	11. $\frac{d}{dx}[\log_a u] = \frac{u'}{(\ln a)u}$	12. $\frac{d}{dx}[a^u] = (\ln a)a^u u'$
13. $\frac{d}{dx}[\sin u] = (\cos u)u'$	14. $\frac{d}{dx}[\cos u] = -(\sin u)u'$	15. $\frac{d}{dx}[\tan u] = (\sec^2 u)u'$
16. $\frac{d}{dx}[\cot u] = -(\csc^2 u)u'$	17. $\frac{d}{dx}[\sec u] = (\sec u \tan u)u'$	18. $\frac{d}{dx}[\csc u] = -(\csc u \cot u)u'$
19. $\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$	20. $\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$	21. $\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2}$
22. $\frac{d}{dx}[\text{arccot } u] = \frac{-u'}{1+u^2}$	23. $\frac{d}{dx}[\text{arcsec } u] = \frac{u'}{\lvert u\rvert\sqrt{u^2-1}}$	24. $\frac{d}{dx}[\text{arccsc } u] = \frac{-u'}{\lvert u\rvert\sqrt{u^2-1}}$