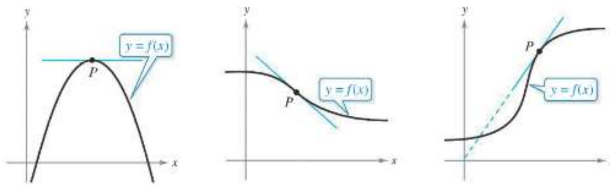
2.1 The Derivative and the Tangent Line Problem

- Find the slope of the tangent line to a curve at a point.
- Use the limit definition to find the derivative of a function.
- Understand the relationship between differentiability and continuity.

Calculus grew out of four major problems that European mathematicians were working on during the seventeenth century.

1. The tangent line problem (Section 1.1 and this section)
2. The velocity and acceleration problem (Sections 2.2 and 2.3)
3. The minimum and maximum problem (Section 3.1)
4. The area problem (Sections 1.1 and 4.2)



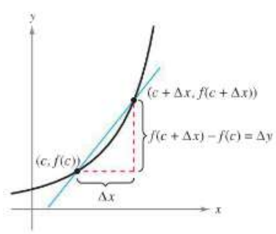
Tangent line to a curve at a point

2.1 The Derivative and the Tangent Line Problem

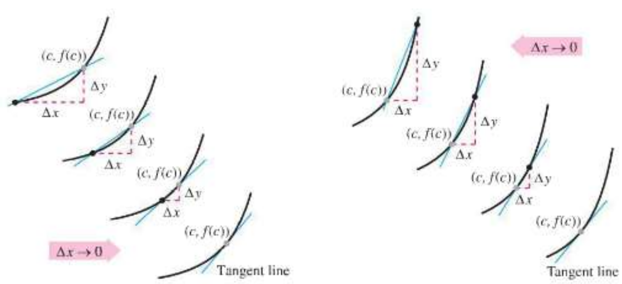
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m_{\text{sec}} = \frac{f(c + \Delta x) - f(c)}{(c + \Delta x) - c} \qquad \frac{\text{Change in } y}{\text{Change in } x}$$

$$m_{\text{sec}} = \frac{f(c + \Delta x) - f(c)}{\Delta x}. \qquad \text{Slope of secant line}$$



The secant line through $(c, f(c))$ and $(c + \Delta x, f(c + \Delta x))$



Tangent line approximations

2.1 The Derivative and the Tangent Line Problem

Definition of Tangent Line with Slope m

If f is defined on an open interval containing c, and if the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = m$$

exists, then the line passing through $(c, f(c))$ with slope m is the **tangent line** to the graph of f at the point $(c, f(c))$.

Example 1: Use the definition of the slope of a tangent line to find the slope of the graph of $f(x) = -5x + 1$ when $c = 3$.

Example 2: Find the slopes of the tangent lines to the graph of $f(x) = x^2 - 2$ at the points $(-3,7)$and $(1,-1)$.

2.1 The Derivative and the Tangent Line Problem

Definition of the Derivative of a Function

The **derivative** of f at x is

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

provided the limit exists. For all x for which this limit exists, f' is a function of x.

Connecting Notations

The notation $f'(x)$ is read as "f prime of x."

$f'(x), \quad \frac{dy}{dx}, \quad y', \quad \frac{d}{dx}[f(x)], \quad D_x[y].$

Notation for derivatives

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x).$$

Example 3: Find the derivative of $f(x) = 4x^2 - 5x$.

2.1 The Derivative and the Tangent Line Problem

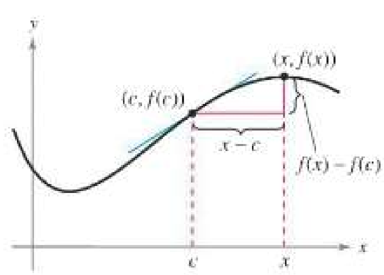
Example 4: Find $f'(x)$ for $f(x) = \sqrt{x} + 1$. Then find the slopes of the graph at f at the points $(4,3)$ and $(9,4)$.

Example 5: Find the derivative with respect to t for the function $y = \frac{1}{2t^2}$. Then find an equation of the tangent line to the graph at the point $\left(1, \frac{1}{2}\right)$.

2.1 The Derivative and the Tangent Line Problem

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Alternative form of derivative



As x approaches c, the secant line approaches the tangent line.

2.1 The Derivative and the Tangent Line Problem

Example 6: Determine whether the function $f(x) = |3x + 1|$ is differentiable at $x = -\frac{1}{3}$.

Example 7: Determine whether the function $f(x) = (x - 2)^{\frac{1}{5}}$ is differentiable $x = 2$.

THEOREM 2.1 Differentiability Implies Continuity
If f is differentiable at $x = c$, then f is continuous at $x = c$.

2.2 Basic Differentiation Rules and Rates of Change

- Find the derivative of a function using the Constant Rule.
- Find the derivative of a function using the Power Rule.
- Find the derivative of a function using the Constant Multiple Rule.
- Find the derivative of a function using the Sum and Difference Rules.
- Find the derivatives of the sine function and of the cosine function.
- Find the derivatives of exponential functions.
- Use derivatives to find rates of change.

THEOREM 2.2 The Constant Rule
The derivative of a constant function is 0. That is, if c is a real number, then

$$\frac{d}{dx}[c] = 0.$$

(See Figure 2.14.)

2.2 Basic Differentiation Rules and Rates of Change

Example 1: Find the derivative of each function.

a. $f(x) = \frac{3}{7}$ b. $g(t) = e^{0.12}$

THEOREM 2.3 The Power Rule

If n is a rational number, then the function $f(x) = x^n$ is differentiable and

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

For f to be differentiable at $x = 0$, n must be a number such that x^{n-1} is defined on an interval containing 0.

Example 2: Use the Power Rule to find the derivative of each function.

a. $f(x) = x^5$ b. $g(x) = \sqrt[4]{x^3}$ c. $y = \frac{1}{x^3}$

2.2 Basic Differentiation Rules and Rates of Change

Example 3: Find the slope of the graph of $f(x) = \frac{1}{x^4}$ for each value of x.

a. $x = -1$ b. $x = 1$ c. $x = 2$

Example 4: Find an equation of the tangent line to the graph of $f(x) = \sqrt[3]{x}$ when $x = 1$.

2.2 Basic Differentiation Rules and Rates of Change

THEOREM 2.4 The Constant Multiple Rule

If f is a differentiable function and c is a real number, then cf is also differentiable and

$$\frac{d}{dx}[cf(x)] = cf'(x).$$

Example 5: Use the Constant Multiple Rule to find the derivative of each function.

a. $f(x) = 2x^7$ b. $g(x) = \frac{3}{x^2}$ c. $y = \frac{\sqrt[6]{x^5}}{8}$

Example 6: Rewrite each function in the form $y = cf(x)$. Then use the Constant Multiple Rule to find the derivative of the function.

a. $y = \frac{3x^4}{8}$ b. $y = \frac{(3x)^4}{8}$ c. $y = \frac{9}{7x^2}$ d. $y = \frac{9}{(7x)^2}$

2.2 Basic Differentiation Rules and Rates of Change

THEOREM 2.5 The Sum and Difference Rules

The sum (or difference) of two differentiable functions f and g is itself differentiable. Moreover, the derivative of $f + g$ (or $f - g$) is the sum (or difference) of the derivatives of f and g.

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$ Sum Rule

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$ Difference Rule

Example 7: Use the Sum and Difference Rules to find the derivatives of each function.

a. $f(x) = -x^4 + 3x - 9$ b. $g(x) = \sqrt{x} + 3x^2$ c. $y = \frac{x^3 - 3x^2 - 5}{x^2}$

2.2 Basic Differentiation Rules and Rates of Change

THEOREM 2.6 Derivatives of Sine and Cosine Functions

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\cos x] = -\sin x$$

Example 8: Find the derivative of each function.

a. $f(x) = 3\cos x$ b. $g(x) = \frac{2\sin x}{3}$ c. $y = \frac{\cos x}{2} + \cos\frac{\pi}{2}$

THEOREM 2.7 Derivative of the Natural Exponential Function

$$\frac{d}{dx}[e^x] = e^x$$

Example 9: Find the derivative of each function.

a. $f(x) = 5e^x$ b. $g(x) = e^x - 4x$ c. $y = 7e^x - \cos x$

2.2 Basic Differentiation Rules and Rates of Change

The function s that gives the position (relative to the origin) of an object as a function of time t is called a **position function.** If, over a period of time Δt, the object changes its position by the amount

$$\Delta s = s(t + \Delta t) - s(t)$$

then, by the familiar formula

$$\text{Rate} = \frac{\text{distance}}{\text{time}}$$

the **average velocity** is

$$\frac{\text{Change in distance}}{\text{Change in time}} = \frac{\Delta s}{\Delta t}. \quad \text{Average velocity}$$

Insight
On the AP® Exam, be sure to include appropriate units in your solution when applicable. You may not earn a point on a free-response question when you fail to include units with your solution.

Example 10: A tennis ball is dropped from a height of 150 feet. The ball's height s at time t is the position function $s = -16t^2 + 150$, where s is measured in feet and t is measured in seconds. Find the average velocity over each of the following intervals.

a. [2,3] b. [2,2.5] c. [2,2.1]

2.2 Basic Differentiation Rules and Rates of Change

$$v(t) = \lim_{\Delta t \to 0} \frac{s(t + \Delta t) - s(t)}{\Delta t} = s'(t).$$ Velocity function

The position of a free-falling object (neglecting air resistance) under the influence of gravity can be represented by the equation

$$s(t) = \frac{1}{2}gt^2 + v_0t + s_0$$ Position function

where s_0 is the initial height of the object, v_0 is the initial velocity of the object, and g is the acceleration due to gravity. On Earth, the value of g is approximately -32 feet per second per second or -9.8 meters per second per second.

Example 11: A water balloon is thrown upward form the top of an 80-foot building with an initial velocity of 64 feet per second. The height s (in feet) of the balloon can be modeled by the position function $s = -16t^2 + 64t + 80$, where t is the time in seconds since it was thrown.

a. How long is the water balloon in the air?

b. What is the velocity of the water balloon when it hits the ground?

2.3 Product and Quotient Rules and Higher-Order Derivatives

- Find the derivative of a function using the Product Rule.
- Find the derivative of a function using the Quotient Rule.
- Find the derivative of a trigonometric function.
- Find a higher-order derivative of a function.

THEOREM 2.8 The Product Rule

The product of two differentiable functions f and g is itself differentiable. Moreover, the derivative of fg is the first function times the derivative of the second, plus the second function times the derivative of the first.

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

Remark

A version of the Product Rule that some people prefer is

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

The advantage of this form is that it generalizes easily to products of three or more factors.

Proof

2.3 Product and Quotient Rules and Higher-Order Derivatives

Example 1: Find the derivatives of each function.

a. $h(x) = (3x+5)(x^2 - 2x^3)$ b. $y = x^2 e^x$ c. $y = 4x^2 \sin x - 7\cos x$

THEOREM 2.9 The Quotient Rule

The quotient f/g of two differentiable functions f and g is itself differentiable at all values of x for which $g(x) \neq 0$. Moreover, the derivative of f/g is given by the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0$$

Remark

From the Quotient Rule, you can see that the derivative of a quotient is not (in general) the quotient of the derivatives.

2.3 Product and Quotient Rules and Higher-Order Derivatives

Example 2: Find the derivatives of each function.

a. $y = \frac{3x+1}{2x^2-5}$ b. $f(x) = \frac{2+\frac{1}{x}}{x-1}$ c. $y = \frac{2(x^3-x^2)}{5x}$

Example 3: For part (b) above, find an equation of the tangent line at $\left(2, \frac{5}{2}\right)$

2.3 Product and Quotient Rules and Higher-Order Derivatives

THEOREM 2.10 Derivatives of Trigonometric Functions

$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$$

Example 4: Find the derivatives of each function.

a. $y = 3x^2 - \csc x$

b. $y = x^3 \cot x$

2.3 Product and Quotient Rules and Higher-Order Derivatives

Higher-Order Derivatives

$s(t)$ Position function

$v(t) = s'(t)$ Velocity function

$a(t) = v'(t) = s''(t)$ Acceleration function

Connecting Notations

The notations y'' and y''' are read as "y double prime" and "y triple prime," respectively. Notice that the prime notation is only used for the first, second, and third derivatives.

First derivative:	y',	$f'(x)$,	$\frac{dy}{dx}$,	$\frac{d}{dx}[f(x)]$,	$D_x[y]$
Second derivative:	y'',	$f''(x)$,	$\frac{d^2y}{dx^2}$,	$\frac{d^2}{dx^2}[f(x)]$,	$D_x^2[y]$
Third derivative:	y''',	$f'''(x)$,	$\frac{d^3y}{dx^3}$,	$\frac{d^3}{dx^3}[f(x)]$,	$D_x^3[y]$
Fourth derivative:	$y^{(4)}$,	$f^{(4)}(x)$,	$\frac{d^4y}{dx^4}$,	$\frac{d^4}{dx^4}[f(x)]$,	$D_x^4[y]$
⋮					
nth derivative:	$y^{(n)}$,	$f^{(n)}(x)$,	$\frac{d^ny}{dx^n}$,	$\frac{d^n}{dx^n}[f(x)]$,	$D_x^n[y]$

2.3 Product and Quotient Rules and Higher-Order Derivatives

Example 5: The position function of an object dropped on Mars is $s(t) = -1.85t^2 + 3$, where $s(t)$ is the height in meters and t is the time in seconds after the object is dropped. What is the ratio of Earth's gravitational force to Mars'?

Note: acceleration due to gravity on Earth is -9.8 meters per second.