1.4 Continuity and One-Sided Limits

- Determine continuity at a point and continuity on an open interval.
- Determine one-sided limits and continuity on a closed interval.
- Use properties of continuity.
- Understand and use the Intermediate Value Theorem.

Definition of Continuity

Continuity at a Point

A function f is **continuous at c** when these three conditions are met.

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$

Continuity on an Open Interval

A function is **continuous on an open interval (a, b)** when the function is continuous at each point in the interval. A function that is continuous on the entire real number line $(-\infty, \infty)$ is **everywhere continuous.**

Insight

You must know the precise definition of continuity for the AP® Exam, and be able to understand which condition of the definition is not met if a function is not continuous at a particular point.

Types of Discontinuity: Removable and Non-Removable.

A discontinuity at *c* is called removable when *f* can be made continuous by appropriately defining (or redefining) *f(c)*.

1.4 Continuity and One-Sided Limits

- Determine continuity at a point and continuity on an open interval.

<u>**Example 1:**</u> Discuss the continuity of each function.

$a. f(x) = \dfrac{2x+1}{x+1}$

$b. y = \dfrac{4}{x^2+1}$

$c. h(x) = \dfrac{x^2+5x+6}{x+3}$

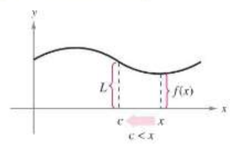
$d. g(x) = \begin{cases} 3x-5, x<1 \\ -2x^2, x \geq 1 \end{cases}$

1.4 Continuity and One-Sided Limits

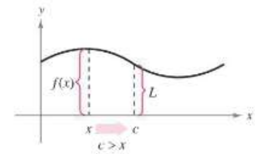
▶ Determine one-sided limits and continuity on a closed interval.

$\lim_{x \to c^+} f(x) = L.$ Limit from the right

$\lim_{x \to c^-} f(x) = L.$ Limit from the left



(a) Limit as x approaches c from the right.



(b) Limit as x approaches c from the left.

$[\![x]\!] =$ greatest integer n such that $n \le x$. Greatest integer function

Example 2: Find the limits of the following functions from the left and right *c*.

$a.\ f(x) = \frac{|x|}{4x}, c = 0$

$b.\ g(x) = [\![x-2]\!], c = 3$

1.4 Continuity and One-Sided Limits

▶ Use properties of continuity.

THEOREM 1.10 The Existence of a Limit

Let f be a function, and let c and L be real numbers. The limit of $f(x)$ as x approaches c is L if and only if

$$\lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

Definition of Continuity on a Closed Interval

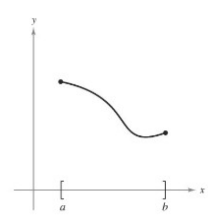
A function f is **continuous on the closed interval $[a, b]$** when f is continuous on the open interval (a, b) and

$$\lim_{x \to a^+} f(x) = f(a)$$

and

$$\lim_{x \to b^-} f(x) = f(b).$$

The function f is **continuous from the right** at a and **continuous from the left** at b. (See Figure 1.30.)



Continuous function on a closed interval

1.4 Continuity and One-Sided Limits

▶ Use properties of continuity.

Example 3: Discuss the continuity of the following.

a. $f(x) = \sqrt{4 - x^2}$

b. $f(x) = \frac{\sin x}{x^2+3}$

c. To ship a package overnight, a delivery services charges \$9 for the first pound and \$1 for each additional pound or portion of a pound. Let x represent the weight of the package and let $f(x)$ represent the shipping cost. Show that the limit of $f(x)$ as x approaches 3 does not exist.

1.4 Continuity and One-Sided Limits

▶ Use properties of continuity.

THEOREM 1.11 Properties of Continuity

If b is a real number and f and g are continuous at $x = c$, then the functions listed below are also continuous at c.

1. Scalar multiple: bf
2. Sum or difference: $f \pm g$
3. Product: fg
4. Quotient: $\frac{f}{g}$, $g(c) \neq 0$

A proof of this theorem is given in Appendix A.

THEOREM 1.12 Continuity of a Composite Function

If g is continuous at c and f is continuous at $g(c)$, then the composite function given by $(f \circ g)(x) = f(g(x))$ is continuous at c.

Remark

One consequence of Theorem 1.12 is that when f and g satisfy the given conditions, you can determine the limit of $f(g(x))$ as x approaches c to be

$$\lim_{x \to c} f(g(x)) = f(g(c)).$$

1.4 Continuity and One-Sided Limits

▶ Use properties of continuity.

Example 4: Describe the intervals(s) on which each function is continuous.

a. $f(x) = \tan\frac{\pi x}{6}$

b. $g(x) = \cos x^2$

1.4 Continuity and One-Sided Limits

▶ Understand and use the Intermediate Value Theorem.

THEOREM 1.13 Intermediate Value Theorem

If f is continuous on the closed interval $[a, b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that

$f(c) = k.$

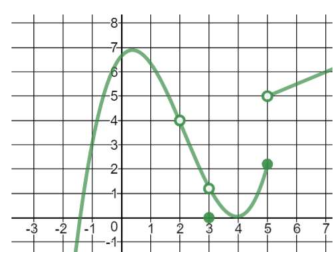
Example 5: The graph of f is shown. For each value of c, find $\lim_{x\to c^-} f(x)$, $\lim_{x\to c^+} f(x)$, and $f(c)$ if possible. Then tell whether f is continuous at $x = c$.

a. $c = 2$

b. $c = 3$

c. $c = 5$

d. $c = 7$



1.5 Infinite Limits

- Determine infinite limits from the left and from the right.
- Find and sketch the vertical asymptotes of the graph of a function.

Definition of Infinite Limits

Let f be a function that is defined at every real number in some open interval containing c (except possibly at c itself). The statement

$$\lim_{x\to c} f(x) = \infty$$

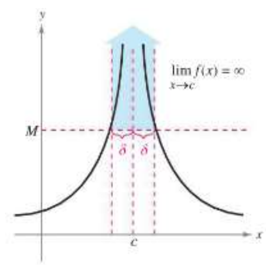
means that for each $M > 0$ there exists a $\delta > 0$ such that $f(x) > M$ whenever $0 < |x - c| < \delta$. (See Figure 1.39.) Similarly, the statement

$$\lim_{x\to c} f(x) = -\infty$$

means that for each $N < 0$ there exists a $\delta > 0$ such that $f(x) < N$ whenever

$$0 < |x - c| < \delta.$$

To define the **infinite limit from the left,** replace $0 < |x - c| < \delta$ by $c - \delta < x < c$. To define the **infinite limit from the right,** replace $0 < |x - c| < \delta$ by $c < x < c + \delta$.



Infinite limits

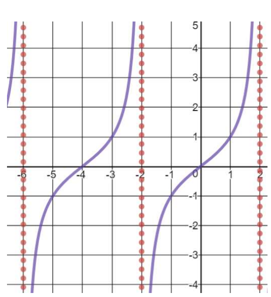
1.5 Infinite Limits

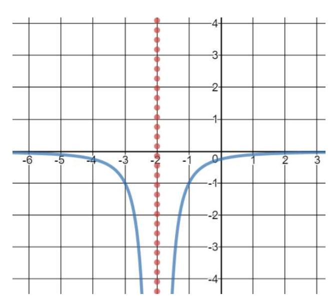
- Determine infinite limits from the left and from the right.

Example 1: Determine the limit of each function as x approaches -2 from the left and from the right.

$a.\ f(x) = \tan\dfrac{\pi x}{4}$

$\text{b.}\ y = -\dfrac{1}{(x+2)^2}$





1.5 Infinite Limits

▶ Find and sketch the vertical asymptotes of the graph of a function.

Definition of Vertical Asymptote

If $f(x)$ approaches infinity (or negative infinity) as x approaches c from the right or the left, then the line $x = c$ is a **vertical asymptote** of the graph of f.

Connecting Concepts

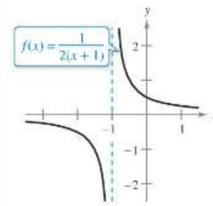
If the graph of a function f has a vertical asymptote at $x = c$, then f is *not continuous* at c.

THEOREM 1.14 Vertical Asymptotes

Let f and g be continuous on an open interval containing c. If $f(c) \neq 0$, $g(c) = 0$, and there exists an open interval containing c such that $g(x) \neq 0$ for all $x \neq c$ in the interval, then the graph of the function

$$h(x) = \frac{f(x)}{g(x)}$$

has a vertical asymptote at $x = c$.
A proof of this theorem is given in Appendix A.



1.5 Infinite Limits

▶ Find and sketch the vertical asymptotes of the graph of a function.

<u>Example 2:</u> Determine all vertical asymptotes of the graphs of the functions.

$a.\ f(x) = \frac{4x}{x-2}$

b. $g(x) = \frac{5x^2}{x^2-1}$

c. $h(x) = \tan\frac{\pi x}{2}$

d. $f(x) = \frac{x^2+x-6}{x^2+4x+3}$

1.5 Infinite Limits

▶ Find and sketch the vertical asymptotes of the graph of a function.

Example 3: Find each limit.

$a.\ \lim_{x\to -2^-} \frac{x}{(x+2)^2}$

$b.\ \lim_{x\to -2^+} \frac{x}{(x+2)^2}$

1.5 Infinite Limits

▶ Find and sketch the vertical asymptotes of the graph of a function.

THEOREM 1.15 Properties of Infinite Limits

Let c and L be real numbers, and let f and g be functions such that

$$\lim_{x\to c} f(x) = \infty \quad \text{and} \quad \lim_{x\to c} g(x) = L.$$

1. Sum or difference: $\lim_{x\to c} [f(x) \pm g(x)] = \infty$
2. Product: $\lim_{x\to c} [f(x)g(x)] = \infty, \quad L > 0$
   $\lim_{x\to c} [f(x)g(x)] = -\infty, \quad L < 0$
3. Quotient: $\lim_{x\to c} \frac{g(x)}{f(x)} = 0$

Similar properties hold for one-sided limits and for functions for which the limit of $f(x)$ as x approaches c is $-\infty$. (See Example 5.)

Example 4: Find each limit.

$a.\ \lim_{x\to 0} \left(-3 + \frac{1}{x^4}\right)$

$b.\ \lim_{x\to 0^+} \frac{x+4}{-\ln x^2}$

$c.\ \lim_{x\to \frac{3}{2}^-} 4\tan\frac{\pi x}{3}$

$d.\ \lim_{x\to 1} \left(x + \frac{-1}{(x-1)^2}\right)$

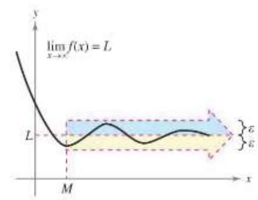
1.6 Limits at Infinity

- Determine (finite) limits at infinity.
- Determine the horizontal asymptotes, if any, of the graph of a function.
- Determine infinite limits at infinity.

Definition of Limits at Infinity

Let L be a real number.

1. The statement $\lim_{x\to\infty} f(x) = L$ means that for each $\varepsilon > 0$ there exists an $M > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x > M$.
2. The statement $\lim_{x\to-\infty} f(x) = L$ means that for each $\varepsilon > 0$ there exists an $N < 0$ such that $|f(x) - L| < \varepsilon$ whenever $x < N$.



$f(x)$ is within ε units of L as $x \to \infty$.

Definition of a Horizontal Asymptote

The line $y = L$ is a **horizontal asymptote** of the graph of f when

$$\lim_{x\to-\infty} f(x) = L \quad \text{or} \quad \lim_{x\to\infty} f(x) = L.$$

THEOREM 1.16 Limits at Infinity

1. If r is a positive rational number and c is any real number, then

$$\lim_{x\to\infty} \frac{c}{x^r} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{c}{x^r} = 0.$$

The second limit is valid only if x^r is defined when $x < 0$.

2. $\lim_{x\to-\infty} e^x = 0$ and $\lim_{x\to\infty} e^{-x} = 0$

A proof of the first part of this theorem is given in Appendix A.

1.6 Limits at Infinity

- Determine (finite) limits at infinity.

Guidelines for Finding Limits at $\pm\infty$ of Rational Functions

1. If the degree of the numerator is *less than* the degree of the denominator, then the limit of the rational function is 0.
2. If the degree of the numerator is *equal to* the degree of the denominator, then the limit of the rational function is the ratio of the leading coefficients.
3. If the degree of the numerator is *greater than* the degree of the denominator, then the limit of the rational function does not exist.

Definition of Infinite Limits at Infinity

Let f be a function defined on the interval (a, ∞).

1. The statement $\lim_{x\to\infty} f(x) = \infty$ means that for each positive number M, there is a corresponding number $N > 0$ such that $f(x) > M$ whenever $x > N$.
2. The statement $\lim_{x\to\infty} f(x) = -\infty$ means that for each negative number M, there is a corresponding number $N > 0$ such that $f(x) < M$ whenever $x > N$.

Similar definitions can be given for the statements

$$\lim_{x\to-\infty} f(x) = \infty \quad \text{and} \quad \lim_{x\to-\infty} f(x) = -\infty.$$

Connecting Concepts

Determining whether a function has an infinite limit at infinity is useful in analyzing the "end behavior" of its graph. You will see examples of this in Section 3.5 on curve sketching.

1.6 Limits at Infinity

- Determine (finite) limits at infinity.
- Determine the horizontal asymptotes, if any, of the graph of a function.
- Determine infinite limits at infinity.

Example 1: Find each limit.

$$a.\ \lim_{x\to-\infty}\left(7-\frac{1}{x^2}\right)$$

$$b.\ \lim_{x\to-\infty}(e^x-6)$$

$$c.\ \lim_{x\to\infty}\frac{3x}{x-1}$$

1.6 Limits at Infinity

- Determine (finite) limits at infinity.
- Determine the horizontal asymptotes, if any, of the graph of a function.
- Determine infinite limits at infinity.

Example 1: Find each limit.

$$d.\ \lim_{x\to\infty}\frac{-x+4}{5x^2+2}$$

$$e.\ \lim_{x\to\infty}\frac{-x^2+4}{5x^2+2}$$

$$f.\ \lim_{x\to\infty}\frac{-x^3+4}{5x^2+2}$$

1.6 Limits at Infinity

- Determine (finite) limits at infinity.
- Determine the horizontal asymptotes, if any, of the graph of a function.
- Determine infinite limits at infinity.

Example 2: Show that he function has different horizontal asymptotes to the left and to the right.

$$f(x) = \frac{|2x|}{3x+1}$$

1.6 Limits at Infinity

- Determine (finite) limits at infinity.
- Determine the horizontal asymptotes, if any, of the graph of a function.
- Determine infinite limits at infinity.

Example 3: Find each limit.

$a.\ \lim\limits_{x\to\infty} \cos 2x$

$b.\ \lim\limits_{x\to\infty} \dfrac{\cos x + 3x}{x}$

$c.\ \lim\limits_{x\to\infty} -x^5$

$d.\ \lim\limits_{x\to-\infty} -x^5$

1.6 Limits at Infinity

- Determine (finite) limits at infinity.
- Determine the horizontal asymptotes, if any, of the graph of a function.
- Determine infinite limits at infinity.

Example 3: Find each limit.

$e.\ \lim_{x\to\infty} \frac{3x^2 + x}{x - 1}$

$f.\ \lim_{x\to-\infty} \frac{3x^2 + x}{x - 1}$