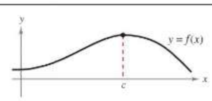
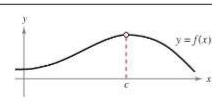
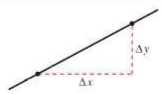
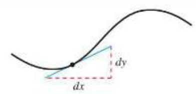
1.1 A Preview of Calculus

- Understand what calculus is and how it compares with precalculus.
- Understand that the tangent line problem is basic to calculus.
- Understand that the area problem is also basic to calculus.

Precalculus mathematics → Limit process → Calculus

Without Calculus		With Differential Calculus	
Value of $f(x)$ when $x = c$		Limit of $f(x)$ as x approaches c	
Slope of a line		Slope of a curve	
Secant line to a curve		Tangent line to a curve	

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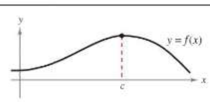
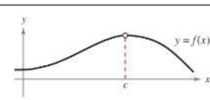
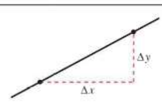
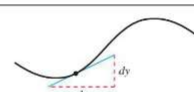
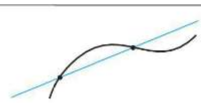
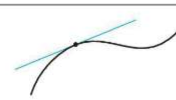
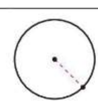
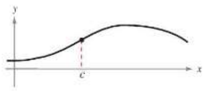
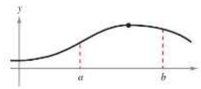
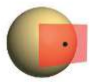
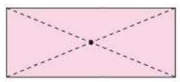
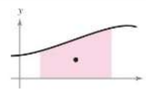
- Understand what calculus is and how it compares with precalculus.

Without Calculus		With Differential Calculus	
Value of $f(x)$ when $x = c$		Limit of $f(x)$ as x approaches c	
Slope of a line		Slope of a curve	
Secant line to a curve		Tangent line to a curve	
Average rate of change between $t = a$ and $t = b$		Instantaneous rate of change at $t = c$	
Curvature of a circle		Curvature of a curve	

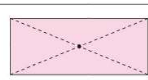
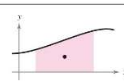
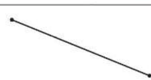
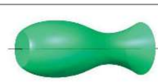
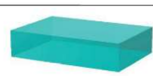
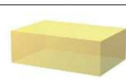
1.1 A Preview of Calculus

▶ Understand what calculus is and how it compares with precalculus.

Without Calculus		With Differential Calculus	
Height of a curve when $x = c$		Maximum height of a curve on an interval	
Tangent plane to a sphere		Tangent plane to a surface	
Direction of motion along a line		Direction of motion along a curve	
Center of a rectangle		Centroid of a region	
Length of a line segment		Length of an arc	

1.1 A Preview of Calculus

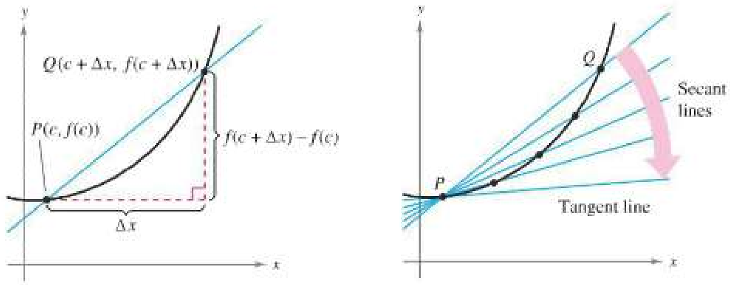
▶ Understand what calculus is and how it compares with precalculus.

Without Calculus		With Integral Calculus	
Center of a rectangle		Centroid of a region	
Length of a line segment		Length of an arc	
Surface area of a cylinder		Surface area of a solid of revolution	
Mass of a solid of constant density		Mass of a solid of variable density	
Volume of a rectangular solid		Volume of a region under a surface	
Sum of a finite number of terms	$a_1 + a_2 + \cdot \cdot \cdot + a_n = S$	Sum of an infinite number of terms	$a_1 + a_2 + a_3 + \cdot \cdot \cdot = S$

1.1 A Preview of Calculus

▶ Understand that the tangent line problem is basic to calculus.

$$m_{sec} = \frac{f(c + \Delta x) - f(c)}{c + \Delta x - c} = \frac{f(c + \Delta x) - f(c)}{\Delta x}.$$

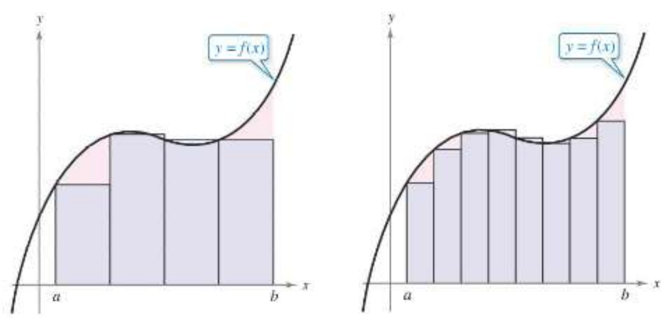


(a) The secant line through $(c, f(c))$ and $(c + \Delta x, f(c + \Delta x))$

(b) As Q approaches P, the secant lines approach the tangent line.

1.1 A Preview of Calculus

▶ Understand that the area problem is also basic to calculus.



Approximation using four rectangles

Approximation using eight rectangles

Figure 1.4

1.2 Finding Limits Graphically and Numerically

- Estimate a limit using a numerical or graphical approach.
- Learn different ways that a limit can fail to exist.
- Study and use a formal definition of limit.

Common Types of Behavior Associated with Nonexistence of a Limit

1. $f(x)$ approaches a different number from the right side of c than it approaches from the left side.
2. $f(x)$ increases or decreases without bound as x approaches c.
3. $f(x)$ oscillates between two fixed values as x approaches c.

Example 1: Use a table or graph to find the following limits.

$$a. \lim_{x\to 0} \frac{x}{\sqrt{x+9}-3}$$

$$b. \lim_{x\to 5} f(x), where\ f(x) = \begin{cases} 1, x \neq 5 \\ -2, x = 5 \end{cases}$$

1.2 Finding Limits Graphically and Numerically

- Learn different ways that a limit can fail to exist.

Example 2: Show that the limit does not exist:

$$a. \lim_{x\to 0} \frac{x}{|x|}$$

$$b. \lim_{x\to 0} \frac{1}{x^4}$$

1.2 Finding Limits Graphically and Numerically

▶ Study and use a formal definition of limit.

Definition of Limit

Let f be a function defined on an open interval containing c (except possibly at c), and let L be a real number. The statement

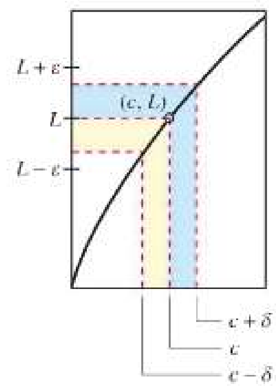
$$\lim_{x \to c} f(x) = L$$

means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if

$$0 < |x - c| < \delta$$

then

$$|f(x) - L| < \varepsilon.$$



The ε-δ definition of the limit of $f(x)$ as x approaches c.

1.2 Finding Limits Graphically and Numerically

Blueprint for proving using the **Formal Definition of a Limit**:

1. Identify $|x - a|$
2. Evaluate $|f(x) - L|$ and simplify till $c|x - a|$
3. Choose $\delta = \frac{\varepsilon}{c}$
4. Summarize if $|x - a| < \delta$ then $|f(x) - L|$ steps $< \varepsilon$

Example 3: Use the $\varepsilon - \delta$ definition of limit to prove that

$a.\ \lim_{x \to -2} (2x + 7) = 3$

$b.\ \lim_{x \to 4} (x^2 - 6) = 10$

1.3 Evaluating Limits Analytically

- Evaluate a limit using properties of limits.
- Develop and use a strategy for finding limits.
- Evaluate a limit using the dividing out technique.
- Evaluate a limit using the rationalizing technique.
- Evaluate a limit using the Squeeze Theorem.

THEOREM 1.1 Some Basic Limits

Let b and c be real numbers, and let n be a positive integer.

1. $\lim_{x\to c} b = b$ **2.** $\lim_{x\to c} x = c$ **3.** $\lim_{x\to c} x^n = c^n$

1.3 Evaluating Limits Analytically

- Evaluate a limit using properties of limits.
- Develop and use a strategy for finding limits.

THEOREM 1.2 Properties of Limits

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the limits $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = K$.

1. Scalar multiple: $\lim_{x\to c} [bf(x)] = bL$

2. Sum or difference: $\lim_{x\to c} [f(x) \pm g(x)] = L \pm K$

3. Product: $\lim_{x\to c} [f(x)g(x)] = LK$

4. Quotient: $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}, \quad K \neq 0$

5. Power: $\lim_{x\to c} [f(x)]^n = L^n$

The proof of Property 1 of this theorem is left as an exercise. (See Exercise 115.) The proofs of the other four properties are given in Appendix A.

1.3 Evaluating Limits Analytically

- Evaluate a limit using properties of limits.
- Develop and use a strategy for finding limits.

THEOREM 1.3 Limits of Polynomial and Rational Functions

If p is a polynomial function and c is a real number, then

$$\lim_{x \to c} p(x) = p(c).$$

If r is a rational function given by $r(x) = p(x)/q(x)$ and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x \to c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

THEOREM 1.4 The Limit of a Function Involving a Radical

Let n be a positive integer. The limit below is valid for all c when n is odd, and is valid for $c > 0$ when n is even.

$$\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$$

A proof of this theorem is given in Appendix A.

1.3 Evaluating Limits Analytically

- Evaluate a limit using properties of limits.
- Develop and use a strategy for finding limits.

THEOREM 1.5 The Limit of a Composite Function

If f and g are functions such that

$$\lim_{x \to c} g(x) = L \quad \text{and} \quad \lim_{x \to L} f(x) = f(L)$$

then

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right) = f(L).$$

A proof of this theorem is given in Appendix A.

THEOREM 1.6 Limits of Transcendental Functions

Let c be a real number in the domain of the given transcendental function.

1. $\lim_{x \to c} \sin x = \sin c$
2. $\lim_{x \to c} \cos x = \cos c$
3. $\lim_{x \to c} \tan x = \tan c$
4. $\lim_{x \to c} \cot x = \cot c$
5. $\lim_{x \to c} \sec x = \sec c$
6. $\lim_{x \to c} \csc x = \csc c$
7. $\lim_{x \to c} a^x = a^c, a > 0$
8. $\lim_{x \to c} \ln x = \ln c$

1.3 Evaluating Limits Analytically

- Evaluate a limit using properties of limits.
- Develop and use a strategy for finding limits.

THEOREM 1.7 Functions That Agree at All but One Point

Let c be a real number, and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c. If the limit of $g(x)$ as x approaches c exists, then the limit of $f(x)$ also exists and

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x).$$

A proof of this theorem is given in Appendix A.

A Strategy for Finding Limits

1. Learn to recognize which limits can be evaluated by direct substitution. (These limits are listed in Theorems 1.1 through 1.6.)
2. When the limit of $f(x)$ as x approaches c *cannot* be evaluated by direct substitution, try to find a function g that agrees with f for all x other than $x = c$. [Choose g such that the limit of $g(x)$ *can* be evaluated by direct substitution.] Then apply Theorem 1.7 to conclude *analytically* that
$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = g(c).$$
3. Use a *graph* or *table* to reinforce your conclusion.

When applying this strategy for finding a limit, remember that some functions do not have a limit (as x approaches c).

1.3 Evaluating Limits Analytically

- Evaluate a limit using properties of limits.
- Develop and use a strategy for finding limits.

Example 1: Find the limits.

$a.\ \lim_{x \to 2} x^3$ $\quad b.\ \lim_{x \to 4} 8x$ $\quad c.\ \lim_{x \to 16} \sqrt[4]{x}$ $\quad d.\ \lim_{x \to 2} (x^2 + 5x + 4)$ $\quad$ e. $\lim_{x \to 2} \left(\frac{x^2+5x+4}{x+4}\right)$

f. $\lim_{x \to -4} \sqrt[3]{2x^2 - 5}$ $\quad$ g. $\lim_{x \to \frac{3\pi}{4}} \tan x$ $\quad h.\ \lim_{x \to 2} e^{3x}$

$i.\ \lim_{x \to -3} f(x)\text{, } where\ f(x) = \begin{cases} x^3 + 2, & x \neq -3 \\ 5, & x = -3 \end{cases}$

1.3 Evaluating Limits Analytically

- Evaluate a limit using the dividing out technique.
- Evaluate a limit using the rationalizing technique.

Example 2: Find the limits.

$$a. \lim_{x\to 2}\left(\frac{x^2+2x-8}{x-2}\right)$$

$$b. \lim_{x\to 0}\frac{\sqrt{x+25}-5}{x}$$

1.3 Evaluating Limits Analytically

- Evaluate a limit using the Squeeze Theorem.

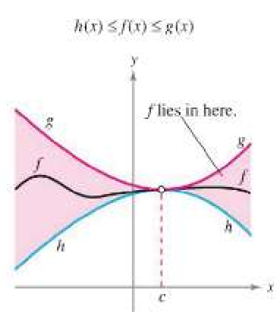
THEOREM 1.8 The Squeeze Theorem

If $h(x) \le f(x) \le g(x)$ for all x in an open interval containing c, except possibly at c itself, and if

$$\lim_{x\to c} h(x) = L = \lim_{x\to c} g(x)$$

then $\lim_{x\to c} f(x)$ exists and is equal to L.

A proof of this theorem is given in Appendix A.



The Squeeze Theorem

Example 3: Show that

$$\lim_{x\to 0} x\sin\frac{1}{x} = 0$$

1.3 Evaluating Limits Analytically

- Develop and use a strategy for finding limits.

THEOREM 1.9 Three Special Limits

1. $\lim_{x\to 0} \frac{\sin x}{x} = 1$ 2. $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$ 3. $\lim_{x\to 0} (1+x)^{1/x} = e$

Remark

The third limit of Theorem 1.9 will be used in Chapter 2 in the development of the formula for the derivative of the exponential function $f(x) = e^x$.

Example 4: Find the limits.

$a. \lim_{x\to 0}\left(\frac{2-2\cos x}{3x}\right)$

$b. \lim_{x\to 0}\left(\frac{\tan 4x}{6x}\right)$