7.1 Basic Integration Rules

Review of Basic Integration Rules ($a > 0$)

1. $\int kf(u)\,du = k\int f(u)\,du$
2. $\int [f(u) \pm g(u)]\,du = \int f(u)\,du \pm \int g(u)\,du$
3. $\int du = u + C$
4. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$
5. $\int \frac{du}{u} = \ln|u| + C$
6. $\int e^u\,du = e^u + C$
7. $\int a^u\,du = \left(\frac{1}{\ln a}\right)a^u + C$
8. $\int \sin u\,du = -\cos u + C$
9. $\int \cos u\,du = \sin u + C$
10. $\int \tan u\,du = -\ln|\cos u| + C$
11. $\int \cot u\,du = \ln|\sin u| + C$
12. $\int \sec u\,du = \ln|\sec u + \tan u| + C$
13. $\int \csc u\,du = -\ln|\csc u + \cot u| + C$
14. $\int \sec^2 u\,du = \tan u + C$
15. $\int \csc^2 u\,du = -\cot u + C$
16. $\int \sec u \tan u\,du = \sec u + C$
17. $\int \csc u \cot u\,du = -\csc u + C$
18. $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\frac{u}{a} + C$
19. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$
20. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\,\text{arcsec}\,\frac{|u|}{a} + C$

7.1 Basic Integration Rules

Example 1: Find each integral

a. $\int \frac{2}{\sqrt{1-x^2}}dx$

b. $\int \frac{2x}{\sqrt{1-x^2}}dx$

c. $\int \frac{2x}{\sqrt{1-x^2}}dx$

7.1 Basic Integration Rules

Example 2: Evaluate $\int_0^4 \frac{x-7}{x^2+16}\,dx$

Example 3: Find $\int \frac{1}{x\sqrt{x^4-4}}\,dx$

Technology

The Trapezoidal Rule can be used to give a good approximation of the value of the integral in Example 2 (for $n = 10$, the approximation is 1.839). When using numerical integration, however, you should be aware that the Trapezoidal Rule does not always give good approximations when one or both of the limits of integration are near a vertical asymptote. For instance, using the Fundamental Theorem of Calculus, you can obtain

$$\int_0^{1.99} \frac{x+3}{\sqrt{4-x^2}}\,dx \approx 6.213.$$

For $n = 10$, the Trapezoidal Rule gives an approximation of 7.500.

7.1 Basic Integration Rules

Example 4:

Evaluate $\int \frac{6}{e^x+6}\,dx$

Example 5: Find $\int (\sin x)e^{-3\,\mathrm{co}} \quad dx$

Procedures for Fitting Integrands to Basic Integration Rules

Technique	*Example*
Expand (numerator).	$(1 + e^x)^2 = 1 + 2e^x + e^{2x}$
Separate numerator.	$\frac{1 + x}{x^2 + 1} = \frac{1}{x^2 + 1} + \frac{x}{x^2 + 1}$
Complete the square.	$\frac{1}{\sqrt{2x - x^2}} = \frac{1}{\sqrt{1 - (x - 1)^2}}$
Divide improper rational function.	$\frac{x^2}{x^2 + 1} = 1 - \frac{1}{x^2 + 1}$
Add and subtract terms in numerator.	$\frac{2x}{x^2 + 2x + 1} = \frac{2x + 2 - 2}{x^2 + 2x + 1} = \frac{2x + 2}{x^2 + 2x + 1} - \frac{2}{(x + 1)^2}$
Use trigonometric identities.	$\cot^2 x = \csc^2 x - 1$
Multiply and divide by Pythagorean conjugate.	$\frac{1}{1 + \sin x} = \left(\frac{1}{1 + \sin x}\right)\left(\frac{1 - \sin x}{1 - \sin x}\right) = \frac{1 - \sin x}{1 - \sin^2 x} = \frac{1 - \sin x}{\cos^2 x} = \sec^2 x - \frac{\sin x}{\cos^2 x}$

<u>Example 6</u>: Find $\int \sin^2 x dx$

7.7 Indeterminate Forms and L'Hôpital's Rule

Indeterminate Form	*Limit*	*Algebraic Technique*
$\frac{0}{0}$	$\lim\limits_{x\to -1} \frac{2x^2 - 2}{x + 1} = \lim\limits_{x\to -1} 2(x - 1) = -4$	Divide numerator and denominator by $(x + 1)$.
$\frac{\infty}{\infty}$	$\lim\limits_{x\to \infty} \frac{3x^2 - 1}{2x^2 + 1} = \lim\limits_{x\to \infty} \frac{3 - (1/x^2)}{2 + (1/x^2)} = \frac{3}{2}$	Divide numerator and denominator by x^2.

THEOREM 7.3 The Extended Mean Value Theorem

If f and g are differentiable on an open interval (a, b) and continuous on $[a, b]$ such that $g'(x) \neq 0$ for any x in (a, b), then there exists a point c in (a, b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

A proof of this theorem is given in Appendix A.

7.7 Indeterminate Forms and L'Hôpital's Rule

THEOREM 7.4 L'Hôpital's Rule

Let f and g be functions that are differentiable on an open interval (a, b) containing c, except possibly at c itself. Assume that $g'(x) \neq 0$ for all x in (a, b), except possibly at c itself. If the limit of $f(x)/g(x)$ as x approaches c produces the indeterminate form $0/0$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is infinite). This result also applies when the limit of $f(x)/g(x)$ as x approaches c produces any one of the indeterminate forms ∞/∞, $(-\infty)/\infty$, $\infty/(-\infty)$, or $(-\infty)/(-\infty)$.

A proof of this theorem is given in Appendix A.

Insight

Indeterminate forms and L'Hôpital's Rule are tested on both versions of the AP® Exam.

7.7 Indeterminate Forms and L'Hôpital's Rule

Example 1: Evaluate $\lim_{x \to 0} \frac{\sin 5x}{2x}$

Example 2: Evaluate $\lim_{x \to \infty} \frac{e^x}{x}$

7.7 Indeterminate Forms and L'Hôpital's Rule

Example 3: Evaluate $\lim_{x \to \infty} \frac{x(\ln x - 1)}{e^x}$

Example 4: Evaluate $\lim_{x \to \infty} xe^{-x/2}$

7.7 Indeterminate Forms and L'Hôpital's Rule

Example 5: Evaluate $\lim_{x\to\infty}(1+e^{-x})^{e^x}$

Example 6: Evaluate $\lim_{x\to 1^-}\left(1-\frac{1}{x}\right)^{1-x}$

7.7 Indeterminate Forms and L'Hôpital's Rule

Example 7: Evaluate $\lim_{x\to 0^+}\left[\frac{1}{x}+\frac{1}{\ln(1-x)}\right]$