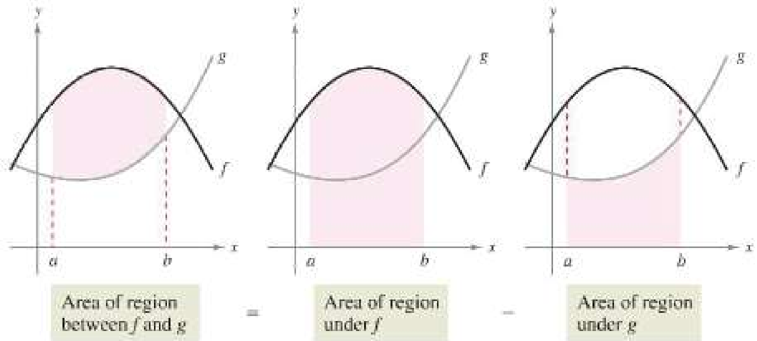


$$\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

Figure 6.2

Area of a Region Between Two Curves

If f and g are continuous on $[a, b]$ and $g(x) \le f(x)$ for all x in $[a, b]$, then the area of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\,dx.$$

Insight

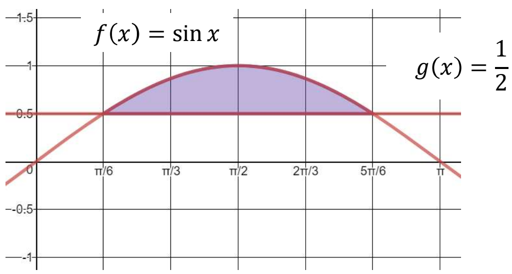
Finding the area of a region between two curves is tested on both the AP® Calculus AB and BC Exams.

<u>Example 1</u>: Find the area of the region bounded by the graphs of $y = x + 3, y = -x^2, x = 1$, and $x = 3$

6.1 Area of a Region Between Two Curves

Example 2: Find the area of the region bounded by the graphs of $f(x) = 3x^2 - 1$ and $g(x) = 3 - x^2$.

Example 3: Find the area of the region.



6.1 Area of a Region Between Two Curves

Example 4: Find the area of the region between the graphs of $f(x) = x^3 - 4x^2$ and $g(x) = 2x^2 - 5x$.

Example 5: Find the area of the region bounded by the graphs of $x = y^2 + 6$ and $x = 2 - 5y$.

Integration as an Accumulation Process

In this section, the integration formula for the area between two curves was developed by using a rectangle as the *representative element.* For each new application in the remaining sections of this chapter, an appropriate representative element will be constructed using precalculus formulas you already know. Each integration formula will then be obtained by summing or accumulating these representative elements.

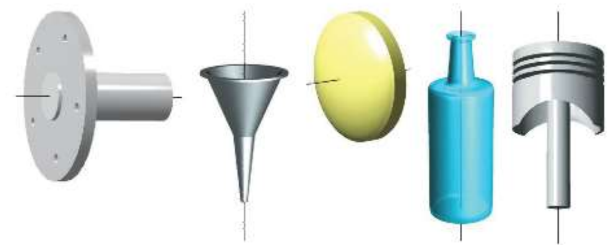
Known precalculus formula ⇒ Representative element ⇒ New integration formula

For example, the area formula in this section was developed as follows.

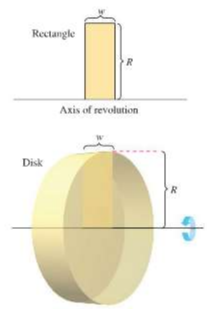
$A = (\text{height})(\text{width})$ ⇒ $\Delta A = [f(x) - g(x)]\,\Delta x$ ⇒ $A = \int_a^b [f(x) - g(x)]\,dx$

Example 6: Find the area of the region bounded by the graph of $y = -2x^2 + x + 1$ and the x-axis. Descirbe the integration as an accumulation process.

6.2 Volume: The Disk and Washer Methods



Solids of revolution
Figure 6.12



When a region in the plane is revolved about a line, the resulting solid is a **solid of revolution.** The line is called the **axis of revolution.** The simplest such solid is a right circular cylinder or **disk**, which is formed by revolving a rectangle about an axis adjacent to one side of the rectangle as seen at the right.

6.2 Volume: The Disk and Washer Methods

Schematically, the disk method looks like this.

Known Precalculus Formula		*Representative Element*		*New Integration Formula*
Volume of disk $V = \pi R^2 w$	⇨	$\Delta V = \pi[R(x_i)]^2 \, \Delta x$	⇨	Solid of revolution $V = \pi \int_a^b [R(x)]^2 \, dx$

A similar formula can be derived when the axis of revolution is vertical.

The Disk Method

To find the volume of a solid of revolution with the **disk method,** use one of the formulas below. (See Figure 6.15.)

Horizontal Axis of Revolution

$$\text{Volume} = V = \pi \int_a^b [R(x)]^2 \, dx$$

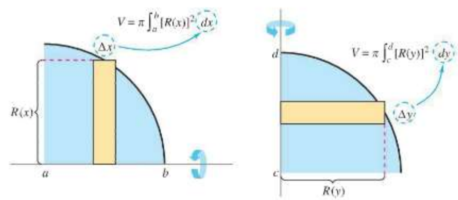
Vertical Axis of Revolution

$$\text{Volume} = V = \pi \int_c^d [R(y)]^2 \, dy$$

The Disk Method

To find the volume of a solid of revolution with the **disk method,** use one of the formulas below. (See Figure 6.15.)

Horizontal Axis of Revolution	*Vertical Axis of Revolution*
Volume $= V = \pi \int_a^b [R(x)]^2\,dx$	Volume $= V = \pi \int_c^d [R(y)]^2\,dy$



Horizontal axis of revolution

Vertical axis of revolution

Figure 6.15

Reasoning

In Figure 6.15, note that you can determine the variable of integration by placing a representative rectangle in the plane region "perpendicular" to the axis of revolution. When the width of the rectangle is Δx, integrate with respect to x, and when the width of the rectangle is Δy, integrate with respect to y.

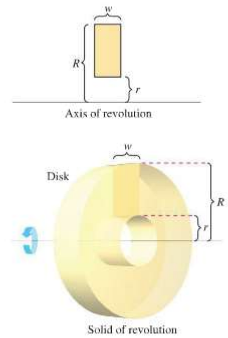
6.2 Volume: The Disk and Washer Methods

Example 1: Find the volume of the solid formed by revolving the region bounded by the graph of $f(x) = -x^2 + 4x - 3$ and the x-axis about the x-axis.

Example 2: Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \sqrt{x},\ y = 1$, and $x = 4$ about the line $y = 1$.

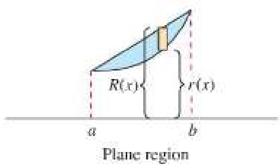
6.2 Volume: The Disk and Washer Methods

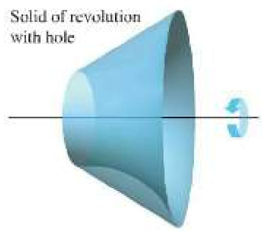
The disk method can be extended to cover solids of revolution with holes by replacing the representative disk with a representative **washer.** The washer is formed by revolving a rectangle about an axis, as shown at the right. If r and R are the inner and outer radii of the washer and w is the width of the washer, then the volume is



Volume of washer $= \pi(R^2 - r^2)$

$$V = \pi\int_a^b \left([R(x)]^2 - [r(x)]^2\right) dx.$$





6.2 Volume: The Disk and Washer Methods

<u>Example 3</u>: Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^2$ and $y = x^3$ about the x-axis.

<u>Example 4</u>: Find the volume of the solid formed by revolving the region bounded by the graphs of $y = 2x + 3$, $y = 0, x = 0$, and $x = 2$ about the y-axis.

Volumes of Solids with Known Cross Sections

1. For cross sections of area $A(x)$ taken perpendicular to the x-axis,

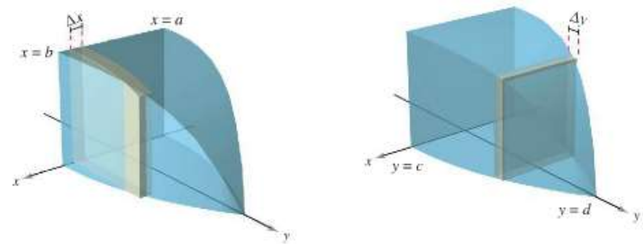
$$\text{Volume} = \int_a^b A(x)\,dx.$$

See Figure 6.24(a).

2. For cross sections of area $A(y)$ taken perpendicular to the y-axis,

$$\text{Volume} = \int_c^d A(y)\,dy.$$

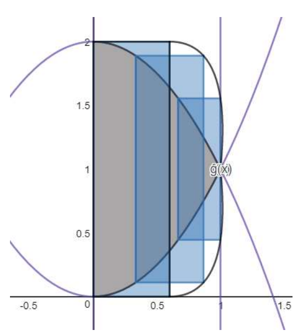
See Figure 6.24(b).



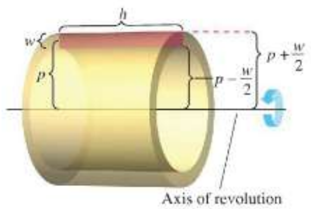
(a) Cross sections perpendicular to x-axis **(b)** Cross sections perpendicular to y-axis

Figure 6.24

Example 5: Find the volume of the solid shown. The base of the solid is the region bounded by the lines $y = x^2$ and $y = 2 - x^2$. The cross sections perpendicular to the x-axis are squares.

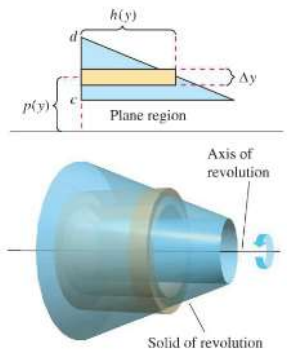


6.3 Volume: The Shell Method



An alternate method for finding volume of a solid of revolution is called the **shell method.** It uses cylindrical shells.

$$\Delta V = 2\pi[p(y)h(y)]\,\Delta y.$$



6.3 Volume: The Shell Method

The Shell Method

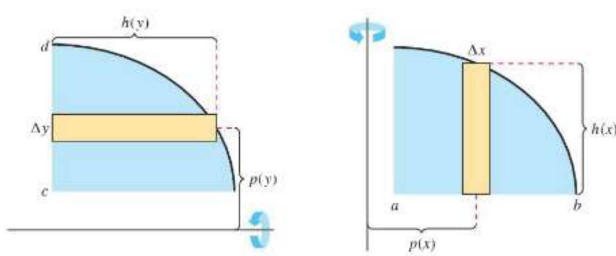
To find the volume of a solid of revolution with the **shell method,** use one of the formulas below. (See Figure 6.29.)

Horizontal Axis of Revolution

$$\text{Volume} = V = 2\pi \int_c^d p(y)h(y)\,dy$$

Vertical Axis of Revolution

$$\text{Volume} = V = 2\pi \int_a^b p(x)h(x)\,dx$$



Horizontal axis of revolution   Vertical axis of revolution

Figure 6.29

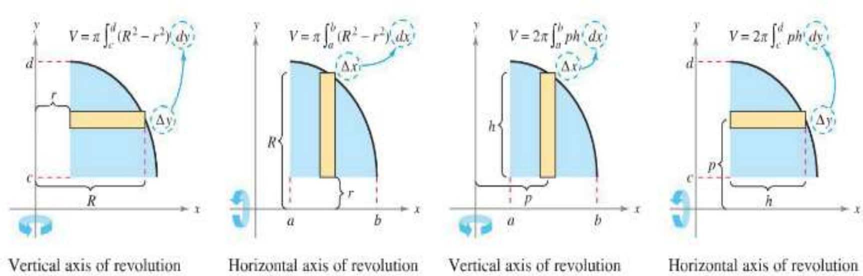
Example 1: Find the volume of the solid of revolution formed by revolving the region bounded by $y = -x^2 + 6x - 5$ and the x-axis $(1 \le x \le 5)$ about the y-axis.

Example 2: Find the volume of the solid of revolution formed by revolving the region bounded by $x = 4 - y^2$ and the y-axis $(0 \le y \le 2)$ about the x-axis.

6.3 Volume: The Shell Method

Comparison of Disk, Washer, and Shell Methods

The disk, washer, and shell methods can be distinguished as follows. For the disk and washer methods, the representative rectangle is always *perpendicular* to the axis of revolution, whereas for the shell method, the representative rectangle is always *parallel* to the axis of revolution. The washer method and the shell method are compared in Figure 6.32.



Washer method: Representative rectangle is perpendicular to the axis of revolution.

Shell method: Representative rectangle is parallel to the axis of revolution.

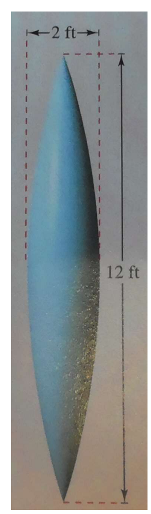
Figure 6.32

6.3 Volume: The Shell Method

Example 3: Use the shell method to find the volume of the solid formed by revolving the region bounded by the graphs of $y = 2x + 3, y = 0, x = 0$, and $x = 2$ about the y-axis.

6.3 Volume: The Shell Method

Example 4: A pontoon is made in the shape shown. The pontoon is designed by rotating the graph of $x = 1 - \frac{y^2}{36}, -6 \le y \le 6$ about the y-axis, where x and y are measured in feet. Find the volume of the pontoon.



6.3 Volume: The Shell Method

Example 5: Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^3 + x^2 + 3, y = 3,$ and $x = 1$ about the line $x = 3$.