4.4 The Fundamental Theorem of Calculus

THEOREM 4.11 The Fundamental Theorem of Calculus

If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Insight

The Fundamental Theorem of Calculus is a foundational concept that is tested throughout the AP® Exam by means of analytic, verbal, numerical, and graphical approaches.

Guidelines for Using the Fundamental Theorem of Calculus

1. *Provided you can find* an antiderivative of f, you now have a way to evaluate a definite integral without having to use the limit of a sum.
2. When applying the Fundamental Theorem of Calculus, the notation shown below is convenient.

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a)$$

For instance, to evaluate $\int_1^3 x^3\,dx$, you can write

$$\int_1^3 x^3\,dx = \frac{x^4}{4}\Big]_1^3 = \frac{3^4}{4} - \frac{1^4}{4} = \frac{81}{4} - \frac{1}{4} = 20.$$

3. It is not necessary to include a constant of integration C in the antiderivative.

$$\int_a^b f(x)\,dx = \Big[F(x) + C\Big]_a^b = [F(b) + C] - [F(a) + C] = F(b) - F(a)$$

4.4 The Fundamental Theorem of Calculus

<u>Example 1</u>: Evaluate each definite integral.

$a. \int_3^4 (1 - x^3)dx$

$b. \int_1^9 \frac{1}{\sqrt{x}}dx$

$c. \int_\pi^{2\pi} -\sin x\,dx$

$d. \int_1^3 |x^2 - 4|dx$

Example 2: Find the area of the region bounded by the graph of $y = \frac{x^3+2}{4}$, the x-axis, and the vertical lines x=0 and x=2.

THEOREM 4.12 Mean Value Theorem for Integrals

If f is continuous on the closed interval $[a, b]$, then there exists a number c in the closed interval $[a, b]$ such that

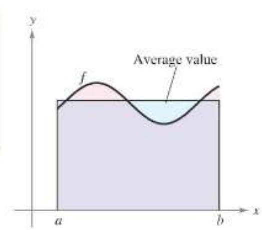
$$\int_a^b f(x)\,dx = f(c)(b-a).$$

Definition of the Average Value of a Function on an Interval

If f is integrable on the closed interval $[a, b]$, then the **average value** of f on the interval is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

See Figure 4.36.



Average value $= \frac{1}{b-a}\int_a^b f(x)\,dx$

Figure 4.36

4.4 The Fundamental Theorem of Calculus

Example 3: Find the average value of $f(x) = 2x^3 + x$ on the interval $[1,2]$.

4.4 The Fundamental Theorem of Calculus

THEOREM 4.13 The Second Fundamental Theorem of Calculus

If f is continuous on an open interval I containing a, then, for every x in the interval,

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x).$$

Insight

The Second Fundamental Theorem of Calculus is frequently tested on the multiple-choice and the free-response sections of the AP® Calculus AB Exam. Remember to apply the Chain Rule when necessary.

Example 4: Evaluate $\frac{d}{dx}\left[\int_0^x \cot t^3\,dt\right]$

Example 5: Find the derivative of $F(x) = \int_{x^2}^{\sin x} \sqrt{t^2+1}\,dt$

4.4 The Fundamental Theorem of Calculus

THEOREM 4.14 The Net Change Theorem

If $F'(x)$ is the rate of change of a quantity $F(x)$, then the definite integral of $F'(x)$ from a to b gives the total change, or **net change,** of $F(x)$ on the interval $[a, b]$.

$$\int_a^b F'(x)\,dx = F(b) - F(a) \qquad \text{Net change of } F$$

Example 6: Water is being drained from a tank at a rate of $(100 + 4t)$ liters per minute, where t is the time in minutes and $0 \le t \le 60$. Find the amount of water that has been drained from the tank in the first 5 minutes.

4.5 Integration by Substitution

THEOREM 4.15 Antidifferentiation of a Composite Function

Let g be a function whose range is an interval I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f on I, then

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C.$$

Letting $u = g(x)$ gives $du = g'(x)\,dx$ and

$$\int f(u)\,du = F(u) + C.$$

Remark

The statement of Theorem 4.15 does not tell how to distinguish between $f(g(x))$ and $g'(x)$ in the integrand. As you become more experienced at integration, your skill in doing this will increase. Of course, part of the key is familiarity with derivatives.

Example 1: Find $\int (2x-3)^3(2)dx$

Example 2: Find $\int \frac{3x^2}{(x^3+4)^2}dx$

4.5 Integration by Substitution

$$\int kf(x)\,dx = k\int f(x)\,dx.$$

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du = F(u) + C.$$

Example 3: Find $\int 3x^2(2x^3+5)dx$

Example 4: Find $\int \frac{e^{1/x}}{x^2}dx$

4.5 Integration by Substitution

$$\int kf(x)\,dx = k\int f(x)\,dx.$$

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du = F(u) + C.$$

Example 5: Find $\int x(3-2x)^4 dx$

Example 6: Find $\int \tan 2x \sec^2 2x\,dx$

Guidelines for Making a Change of Variables

1. Choose a substitution $u = g(x)$. Usually, it is best to choose the *inner* part of a composite function, such as a quantity raised to a power.
2. Compute $du = g'(x)\,dx$.
3. Rewrite the integral in terms of the variable u.
4. Find the resulting integral in terms of u.
5. Replace u by $g(x)$ to obtain an antiderivative in terms of x.
6. Check your answers by differentiating.

THEOREM 4.16 The General Power Rule for Integration

If g is a differentiable function of x, then

$$\int [g(x)]^n g'(x)\,dx = \frac{[g(x)]^{n+1}}{n+1} + C, \quad n \neq -1.$$

Equivalently, if $u = g(x)$, then

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1.$$

THEOREM 4.17 Change of Variables for Definite Integrals

If the function $u = g(x)$ has a continuous derivative on the closed interval $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Example 7: Find $\displaystyle\int_0^3 2x\sqrt{x^2+3}dx$

Example 8: Find $\displaystyle\int_1^9 \frac{x}{(7x+1)^{\frac{2}{3}}}dx$

4.5 Integration by Substitution

THEOREM 4.18 Integration of Even and Odd Functions

Let f be integrable on the closed interval $[-a, a]$.

1. If f is an *even* function, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.
2. If f is an *odd* function, then $\int_{-a}^{a} f(x)\,dx = 0$.

Example 9: Find $\int_{-2}^{2} x^2(3 - x^2)dx$

4.6 The Natural Logarithmic Function: Integration

THEOREM 4.19 Log Rule for Integration

Let u be a differentiable function of x.

1. $\int \frac{1}{x}\,dx = \ln|x| + C$
2. $\int \frac{1}{u}\,du = \ln|u| + C$

$$\int \frac{u'}{u}\,dx = \ln|u| + C.$$

Example 1: Find $\int \frac{3}{x}dx$

Example 2: Find $\int \frac{1}{2-5x}dx$

4.6 The Natural Logarithmic Function: Integration

Example 3: Find each indefinite integral

$a. \int \frac{2x-7}{x^2-7x} dx$

$b. \int \frac{2\cos 2x}{\sin 2x} dx$

$c. \int \frac{2x^2-3}{2x^3-9x} dx$

$d. \int \frac{1}{2-x} dx$

4.6 The Natural Logarithmic Function: Integration

Example 4: Use long division or change of variables to find each indefinite integral

$a. \int \frac{2x^2+x-2}{2x+1} dx$

$b. \int \frac{x}{(2-3x)^2} dx$

Guidelines for Integration

1. Learn a basic list of integration formulas. (By the end of Section 4.7, you will have 20 basic rules.)
2. Find an integration formula that resembles all or part of the integrand, and, by trial and error, find a choice of u that will make the integrand conform to the formula.
3. When you cannot find a u-substitution that works, try altering the integrand. You might try a trigonometric identity, multiplication and division by the same quantity, addition and subtraction of the same quantity, or long division. Be creative.
4. If you have access to computer software that will find antiderivatives symbolically, use it.

Example 5: Solve the differential equation. $\dfrac{dy}{dx} = \dfrac{3x^2}{x^3 - 2}$

Example 6: Find $\displaystyle\int -\cot x \, dx$

4.6 The Natural Logarithmic Function: Integration

Integrals of the Six Basic Trigonometric Functions

$$\int \sin u\,du = -\cos u + C \qquad \int \cos u\,du = \sin u + C$$

$$\int \tan u\,du = -\ln|\cos u| + C \qquad \int \cot u\,du = \ln|\sin u| + C$$

$$\int \sec u\,du = \ln|\sec u + \tan u| + C \qquad \int \csc u\,du = -\ln|\csc u + \cot u| + C$$

Connecting Notations

Using trigonometric identities and properties of logarithms, you could rewrite these six integration rules in other forms. For instance, you could write

$$\int \csc u\,du$$

$$= \ln|\csc u - \cot u| + C.$$

(See Exercises 89–92.)

Example 7: Find the following: $\int_{\pi/6}^{\pi/2} \sqrt{\csc^2 x - 1}dx$

4.7 Inverse Trigonometric Functions: Integration

THEOREM 4.20 Integrals Involving Inverse Trigonometric Functions

Let u be a differentiable function of x, and let $a > 0$.

1. $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C$
2. $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + C$
3. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \text{arcsec} \frac{|u|}{a} + C$

Insight

Integrals that result in inverse trigonometric functions are rarely tested on the AP® Exam. When they are tested, the inverse sine and inverse tangent functions are the most likely topics.

Example 1: Find each indefinite integral.

$a. \int \frac{dx}{x\sqrt{x^2-1}}$

$b. \int \frac{dx}{\sqrt{16-3x^2}}$

$c. \int \frac{dx}{4+25x^2}$

Technology Pitfall

A symbolic integration utility can be useful for integrating functions such as the one in Example 2. In some cases, however, the utility may fail to find an antiderivative for two reasons. First, some elementary functions do not have antiderivatives that are elementary functions. Second, every utility has limitations—you might have entered a function that the utility was not programmed to handle. You should also remember that antiderivatives involving trigonometric functions or logarithmic functions can be written in many different forms. For instance, one utility found the integral in Example 2 to be

$$\int \frac{dx}{\sqrt{e^{2x}-1}} = \arctan \sqrt{e^{2x}-1} + C.$$

Try showing that this antiderivative is equivalent to the one found in Example 2.

4.7 Inverse Trigonometric Functions: Integration

Example 2: Find each of the following.

$a. \int \frac{dx}{\sqrt{e^{4x}-9}}$

$b. \int \frac{x^5+5x^2}{x^6+1}dx$

$c. \int \frac{dx}{x^2-6x+13}$

4.7 Inverse Trigonometric Functions: Integration

Basic Integration Rules ($a > 0$)

1. $\int kf(u)\,du = k\int f(u)\,du$
2. $\int [f(u) \pm g(u)]\,du = \int f(u)\,du \pm \int g(u)\,du$
3. $\int du = u + C$
4. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$
5. $\int \frac{du}{u} = \ln|u| + C$
6. $\int e^u\,du = e^u + C$
7. $\int a^u\,du = \left(\frac{1}{\ln a}\right)a^u + C$
8. $\int \sin u\,du = -\cos u + C$
9. $\int \cos u\,du = \sin u + C$
10. $\int \tan u\,du = -\ln|\cos u| + C$
11. $\int \cot u\,du = \ln|\sin u| + C$
12. $\int \sec u\,du = \ln|\sec u + \tan u| + C$
13. $\int \csc u\,du = -\ln|\csc u + \cot u| + C$
14. $\int \sec^2 u\,du = \tan u + C$
15. $\int \csc^2 u\,du = -\cot u + C$
16. $\int \sec u \tan u\,du = \sec u + C$
17. $\int \csc u \cot u\,du = -\csc u + C$
18. $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C$
19. $\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$
20. $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\text{arcsec}\frac{|u|}{a} + C$

4.7 Inverse Trigonometric Functions: Integration

Example 3: Find each of the following with the techniques and formulas you have learned so far.

$a. \int \frac{1}{4x^2 + 9} dx$

$b. \int \frac{1}{4x^2 - 9} dx$

$c. \int \frac{x}{4x^2 + 9} dx$

4.7 Inverse Trigonometric Functions: Integration

Example 4: Find each of the following with the techniques and formulas you have learned so far.

$a. \int e^{3x} dx$

$b. \int \frac{1}{3x^2} e^{3/x} dx$

$c. \int e^{x^3} dx$