Definition of Antiderivative

A function F is an **antiderivative** of f on an interval I when $F'(x) = f(x)$ for all x in I.

THEOREM 4.1 Representation of Antiderivatives

If F is an antiderivative of f on an interval I, then G is an antiderivative of f on the interval I if and only if G is of the form $G(x) = F(x) + C$ for all x in I where C is a constant.

Example 1: Find the general solution of the differential equation $y' = 2$

4.1 Antiderivatives and Indefinite Integration

Basic Integration Rules

Differentiation Formula	*Integration Formula*
$\frac{d}{dx}[C] = 0$	$\int 0\,dx = C$
$\frac{d}{dx}[kx] = k$	$\int k\,dx = kx + C$
$\frac{d}{dx}[kf(x)] = kf'(x)$	$\int kf(x)\,dx = k\int f(x)\,dx$
$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$	$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
$\frac{d}{dx}[x^n] = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$ Power Rule
$\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x\,dx = \sin x + C$
$\frac{d}{dx}[\cos x] = -\sin x$	$\int \sin x\,dx = -\cos x + C$
$\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
$\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$
$\frac{d}{dx}[\cot x] = -\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$
$\frac{d}{dx}[\csc x] = -\csc x \cot x$	$\int \csc x \cot x\,dx = -\csc x + C$
$\frac{d}{dx}[e^x] = e^x$	$\int e^x\,dx = e^x + C$
$\frac{d}{dx}[a^x] = (\ln a)a^x$	$\int a^x\,dx = \left(\frac{1}{\ln a}\right)a^x + C$
$\frac{d}{dx}[\ln x] = \frac{1}{x}, x > 0$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$

4.1 Antiderivatives and Indefinite Integration

<u>Example 2</u>: Describing antiderivatives and rewrite before integrating

a. $\int 3x dx$

b. $\int \frac{1}{x^3} dx$

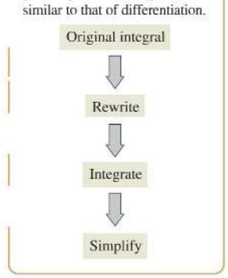
c. $\int \sqrt{x} dx$

d. $\int 2 \sin x\, dx$

e. $\int \frac{3}{x} dx$

Connecting Processes

In Example 2, note that the general pattern of integration is similar to that of differentiation.



4.1 Antiderivatives and Indefinite Integration

Example 3: Integrate polynomials

a. $\int (x+2)dx$

b. $\int (3x^4 - 5x^2 + x)dx$

4.1 Antiderivatives and Indefinite Integration

Example 4: Rewrite before integrating

a. $\int \frac{x+1}{\sqrt{x}} dx$

b. $\int (t^2+1)^2 dt$

4.1 Antiderivatives and Indefinite Integration

Example 5: Finding a particular solution

Find the general solution of $F'(x) = e^x$ and find the particular solution that satisfies the initial condition $F(0) = 3$

4.2 Area

Sigma Notation

The sum of n terms $a_1, a_2, a_3, \ldots, a_n$ is written as

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

where i is the **index of summation,** a_i is the ***i*th term** of the sum, and the **upper and lower bounds of summation** are n and 1.

Example 1: Examples of Sigma Notation

a. $\sum_{i=1}^{6} i$	
b. $\sum_{i=0}^{5} (i+1)$	
c. $\sum_{i=5}^{8} \frac{1}{i^2}$	
d. $\sum_{k=1}^{n} \frac{1}{n}(3k-7)$	

4.2 Area

THEOREM 4.2 Summation Formulas

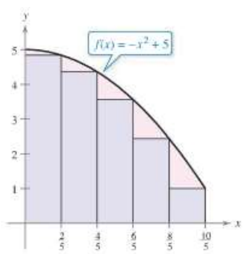
1. $\sum_{i=1}^{n} c = cn$, c is a constant
2. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
3. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
4. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

A proof of this theorem is given in Appendix A.

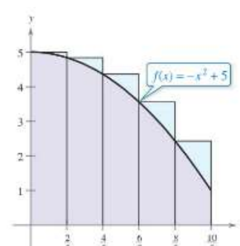
<u>**Example 2**</u>: Evaluate $\sum_{i=1}^{n} \frac{i^2+3}{n^3}$ for n=10, 100, 1000, and 10,000

4.2 Area

<u>**Example 3**</u>: Use the five rectangles in the figure 4.8(a) and (b) to find *two* approximations of the area of the region lying between the graph of $f(x) = -x^2 + 5$ and the x-axis between x=0 and x=2.



(a) The area of the parabolic region is greater than the area of the rectangles.



(b) The area of the parabolic region is less than the area of the rectangles.

Figure 4.8

4.2 Area

Example 4: Find the upper and lower sums for the region bounded by the graph of $f(x) = \frac{1}{4}x^2$ and the x-axis between x=0 and x=2.

4.2 Area

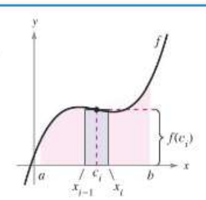
Definition of the Area of a Region in the Plane

Let f be continuous and nonnegative on the interval $[a, b]$. (See Figure 4.13.) The area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x$$

where $x_{i-1} \le c_i \le x_i$ and

$$\Delta x = \frac{b-a}{n}.$$



The width of the ith subinterval is $\Delta x = x_i - x_{i-1}$.
Figure 4.13

THEOREM 4.3 Limits of the Lower and Upper Sums

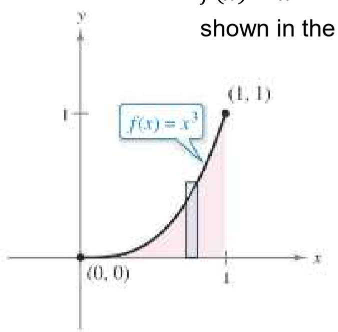
Let f be continuous and nonnegative on the interval $[a, b]$. The limits as $n \to \infty$ of both the lower and upper sums exist and are equal to each other. That is,

$$\lim_{n\to\infty} s(n) = \lim_{n\to\infty} \sum_{i=1}^{n} f(m_i)\,\Delta x$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(M_i)\,\Delta x$$
$$= \lim_{n\to\infty} S(n)$$

where $\Delta x = (b - a)/n$ and $f(m_i)$ and $f(M_i)$ are the minimum and maximum values of f on the subinterval.

4.2 Area

Example 5: Find the area of the region bounded by the graph of $f(x) = x^3$ and the x-axis, and the vertical lines x=0 and x=1, as shown in the figure.



4.2 Area

Example 6: Find the area of the region bounded by the graph of $f(y) = y^3$ and the y-axis for $0 \leq y \leq 1$.

4.2 Area

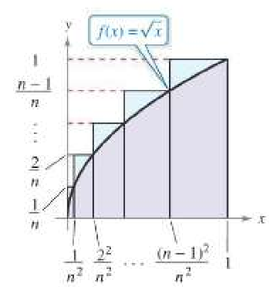
Example 7: Use the Midpoint Rule with $n = 4$ to approximate the area of the region bounded by the graph of $f(x) = x^2 + 4$ and the x-axis for $-2 \le x \le 2$.

4.3 Riemann Sums and Definite Integrals

Example 1: Consider the region bounded by the graph of $f(x) = \sqrt{x}$ and the x-axis for $0 \le x \le 1$, as shown in the figure. Evaluate the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

where c_i is the right endpoint of the partition given by $c_i = \frac{i^2}{n^2}$ and Δx_i is the width of the i^{th} interval.



The subintervals do not have equal widths.
Figure 4.18

Definition of Riemann Sum

Let f be defined on the closed interval $[a, b]$, and let Δ be a partition of $[a, b]$ given by

$$a = x_0 < x_1 < x_2 < \cdot \cdot \cdot < x_{n-1} < x_n = b$$

where Δx_i is the width of the ith subinterval

$$[x_{i-1}, x_i]. \quad \text{ith subinterval}$$

If c_i is *any* point in the ith subinterval, then the sum

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i, \quad x_{i-1} \le c_i \le x_i$$

is called a **Riemann sum** of f for the partition Δ. (The sums in Section 4.2 are examples of Riemann sums, but there are more general Riemann sums than those covered there.)

4.3 Riemann Sums and Definite Integrals

Definition of Definite Integral

If f is defined on the closed interval $[a, b]$ and the limit of Riemann sums over partitions Δ

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

exists (as described above), then f is said to be **integrable** on $[a, b]$ and the limit is denoted by

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i = \int_a^b f(x)\,dx.$$

The limit is called the **definite integral** of f from a to b. The number a is the **lower limit** of integration, and the number b is the **upper limit** of integration.

THEOREM 4.4 Continuity Implies Integrability

If a function f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$. That is, $\int_a^b f(x)\,dx$ exists.

Remark

Later in this chapter, you will learn convenient methods for calculating $\int_a^b f(x)\,dx$ for continuous functions. For now, you must use the limit definition.

Exploration

The Converse of Theorem 4.4 Is the converse of Theorem 4.4 true? That is, when a function is integrable, does it have to be continuous? Explain your reasoning and give examples.

Describe the relationships among continuity, differentiability, and integrability. Which is the strongest condition? Which is the weakest? Which conditions imply other conditions?

4.3 Riemann Sums and Definite Integrals

Example 2: Evaluate the definite integral. $\int_{-2}^{1} 2x\,dx$

THEOREM 4.5 The Definite Integral as the Area of a Region

If f is continuous and nonnegative on the closed interval $[a, b]$, then the area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is

$$\text{Area} = \int_a^b f(x)\,dx.$$

(See Figure 4.22.)

4.3 Riemann Sums and Definite Integrals

Example 3: Sketch the region corresponding to each definite integral. Then evaluate each integral using a geometric formula.

$a.\ \int_{-1}^{7} 7\,dx$

$b.\ \int_{2}^{5} (3x - 6)\,dx$

$c.\ \int_{0}^{4} \sqrt{16 - x^2}\,dx$

4.3 Riemann Sums and Definite Integrals

Example 4: Evaluate each definite integral.

a. $\int_{3}^{3} \sqrt{x^2 - 4}dx$

b. $\int_{2}^{-2} (2 - x)dx$

c. $\int_{-5}^{0} |x + 3|dx$

Definitions of Two Special Definite Integrals

1. If f is defined at $x = a$, then $\int_{a}^{a} f(x)\,dx = 0$.

2. If f is integrable on $[a, b]$, then $\int_{b}^{a} f(x)\,dx = -\int_{a}^{b} f(x)\,dx$.

THEOREM 4.6 Additive Interval Property

If f is integrable on the three closed intervals determined by a, b, and c, then

$$\int_{a}^{b} f(x)\,dx = \int_{a}^{c} f(x)\,dx + \int_{c}^{b} f(x)\,dx.$$

THEOREM 4.7 Properties of Definite Integrals

If f and g are integrable on $[a, b]$ and k is a constant, then the functions kf and $f \pm g$ are integrable on $[a, b]$, and

1. $\int_{a}^{b} kf(x)\,dx = k\int_{a}^{b} f(x)\,dx$ 2. $\int_{a}^{b} [f(x) \pm g(x)]\,dx = \int_{a}^{b} f(x)\,dx \pm \int_{a}^{b} g(x)\,dx.$

4.3 Riemann Sums and Definite Integrals

Remark

Property 2 of Theorem 4.7 can be extended to cover any finite number of functions. (See Example 6.)

Example 5: Evaluate each definite integral.

$\int_{2}^{4} (x^3 - 3x + 7)dx$ $\int_{2}^{4} x^3 dx = 60$ $\int_{2}^{4} x dx = 6$ $\int_{2}^{4} dx = 2$

Example 6: Integral a function with a discontinuity. $\int_{0}^{4} f(x)dx$

$$f(x) = \begin{cases} x + 4, & x < 2 \\ -2x + 16, & x \geq 2 \end{cases}$$

4.3 Riemann Sums and Definite Integrals

THEOREM 4.8 Preservation of Inequality

1. If f is integrable and nonnegative on the closed interval $[a, b]$, then

$$0 \leq \int_a^b f(x)\,dx.$$

2. If f and g are integrable on the closed interval $[a, b]$ and $f(x) \leq g(x)$ for every x in $[a, b]$, then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

A proof of this theorem is given in Appendix A.

THEOREM 4.9 The Trapezoidal Rule

Let f be continuous on $[a, b]$. The Trapezoidal Rule for approximating $\int_a^b f(x)\,dx$ is

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)].$$

Moreover, as $n \to \infty$, the right-hand side approaches $\int_a^b f(x)\,dx$.

Remark

Observe that the coefficients in the Trapezoidal Rule have the following pattern.

1 2 2 2 . . . 2 2 1

Example 7: Use the Trapezoid Rule to approximate $\int_3^7 \sqrt{x-3}\,dx$ for n=4 and n=8