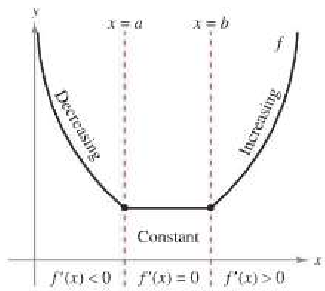
3.3 Increasing and Decreasing Functions and the First Derivative Test

Definitions of Increasing and Decreasing Functions

A function f is **increasing** on an interval when, for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

A function f is **decreasing** on an interval when, for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) > f(x_2)$.



The derivative is related to the slope of a function.

Figure 3.16

Communicating

When discussing the behavior of a function, remember to examine the graph of the function from left to right.

3.3 Increasing and Decreasing Functions and the First Derivative Test

THEOREM 3.5 Test for Increasing and Decreasing Functions

Let f be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b).

1. If $f'(x) > 0$ for all x in (a, b), then f is increasing on $[a, b]$.
2. If $f'(x) < 0$ for all x in (a, b), then f is decreasing on $[a, b]$.
3. If $f'(x) = 0$ for all x in (a, b), then f is constant on $[a, b]$.

Reasoning

The conclusions in the first two cases of Theorem 3.5 are valid even when $f'(x) = 0$ at a finite number of x-values in (a, b).

Example 1: Find the open intervals on which $f(x) = x^3 - \frac{3}{2}x^2$ is increasing or decreasing.

3.3 Increasing and Decreasing Functions and the First Derivative Test

Guidelines for Finding Intervals on Which a Function Is Increasing or Decreasing

Let f be continuous on the interval (a, b). To find the open intervals on which f is increasing or decreasing, use the following steps.

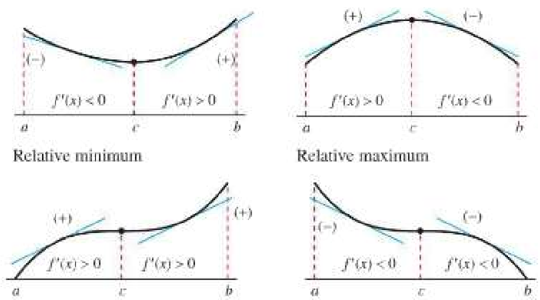
1. Locate the critical numbers of f in (a, b), and use these numbers to determine test intervals.
2. Determine the sign of $f'(x)$ at one test value in each of the intervals.
3. Use Theorem 3.5 to determine whether f is increasing or decreasing on each interval.

These guidelines are also valid when the interval (a, b) is replaced by an interval of the form $(-\infty, b)$, (a, ∞), or $(-\infty, \infty)$.

THEOREM 3.6 The First Derivative Test

Let c be a critical number of a function f that is continuous on an open interval I containing c. If f is differentiable on the interval, except possibly at c, then $f(c)$ can be classified as follows.

1. If $f'(x)$ changes from negative to positive at c, then f has a *relative minimum* at $(c, f(c))$.
2. If $f'(x)$ changes from positive to negative at c, then f has a *relative maximum* at $(c, f(c))$.
3. If $f'(x)$ is positive on both sides of c or negative on both sides of c, then $f(c)$ is neither a relative minimum nor a relative maximum.



Relative minimum

Relative maximum

Neither relative minimum nor relative maximum

Example 2: Find the relative extrema of $f(x) = \frac{1}{2}x - \sin x$ in the interval $(0, 2\pi)$.

Example 3: Find the relative extrema of $f(x) = (x^2 - 4)^{2/3}$.

3.3 Increasing and Decreasing Functions and the First Derivative Test

Example 4: Find the relative extrema of $f(x) = \frac{x^4+1}{x^2}$.

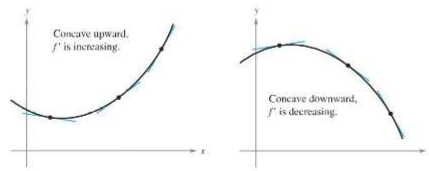
3.3 Increasing and Decreasing Functions and the First Derivative Test

Example 5: Neglecting air resistance, the path of a projectile that is propelled at an angle θ is $y = \frac{g\sec^2\theta}{2{v_0}^2}x^2 + (\tan\theta)x + h,\ 0 \le \theta \le \frac{\pi}{2}$
Where y is the height, x is the horizontal distance, g is the acceleration due to gravity, v_0 is the initial velocity, and h is the initial height. Let $g = -32$ feet per second per second, $v_0 = 24$ feet per second, and $h = 9$ feet. What value of θ will produce a maximum horizontal distance?

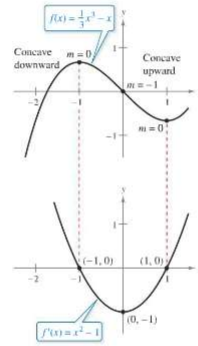
3.4 Concavity and the Second Derivative Test

Definition of Concavity

Let f be differentiable on an open interval I. The graph of f is **concave upward** on I when f' is increasing on the interval and **concave downward** on I when f' is decreasing on the interval.



(a) The graph of f lies above its tangent lines. (b) The graph of f lies below its tangent lines.
Figure 3.24



THEOREM 3.7 Test for Concavity

Let f be a function whose second derivative exists on an open interval I.

1. If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.
2. If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

A proof of this theorem is given in Appendix A.

Reasoning

A third case of Theorem 3.7 could be that if $f''(x) = 0$ for all x in I, then f is linear. Note, however, that concavity is not defined for a line. In other words, a straight line is neither concave upward nor concave downward.

To apply Theorem 3.7, locate the x-values at which $f''(x) = 0$ or f'' does not exist. Use these x-values to determine test intervals. Fi the test intervals.

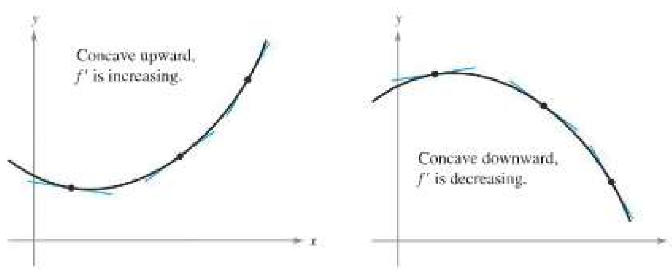
Definition of Point of Inflection

Let f be a function that is continuous on an open interval, and let c be a point in the interval. If the graph of f has a tangent line at this point $(c, f(c))$, then this point is a **point of inflection** of the graph of f when the concavity of f changes from upward to downward (or downward to upward) at the point.

3.4 Concavity and the Second Derivative Test

Definition of Concavity

Let f be differentiable on an open interval I. The graph of f is **concave upward** on I when f' is increasing on the interval and **concave downward** on I when f' is decreasing on the interval.



(a) The graph of f lies above its tangent lines. (b) The graph of f lies below its tangent lines.
Figure 3.24

THEOREM 3.7 Test for Concavity

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A third case of Theorem 3.7 could be that if $f''(x) = 0$ for all x in I, then f is linear. Note, however, that concavity is not defined for a line. In other words, a straight line is neither concave upward nor concave downward.

Example 1: Determine the open intervals on which the graphs are concave upward or concave downward.

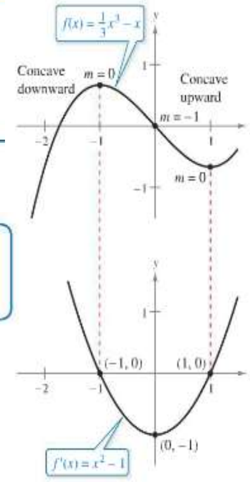
a. $f(x) = e^{-x^2/2}$.

b. $f(x) = \frac{x^2+1}{x^2-4}$.

3.4 Concavity and the Second Derivative Test

Definition of Point of Inflection

Let f be a function that is continuous on an open interval, and let c be a point in the interval. If the graph of f has a tangent line at this point $(c, f(c))$, then this point is a **point of inflection** of the graph of f when the concavity of f changes from upward to downward (or downward to upward) at the point.



The concavity of f is related to the slope of the derivative.

Figure 3.25

THEOREM 3.8 Points of Inflection

If $(c, f(c))$ is a point of inflection of the graph of f, then either $f''(c) = 0$ or f'' does not exist at $x = c$.

3.4 Concavity and the Second Derivative Test

Example 2: Determine the points of inflection and discuss the concavity of the graph of $f(x) = x^4 - 4x^3$

THEOREM 3.9 Second Derivative Test

Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an open interval containing c.

1. If $f''(c) > 0$, then f has a relative minimum at $(c, f(c))$.
2. If $f''(c) < 0$, then f has a relative maximum at $(c, f(c))$.

If $f''(c) = 0$, then the test fails. That is, f may have a relative maximum, a relative minimum, or neither. In such cases, you can use the First Derivative Test.

Example 3: Find the relative extrema of $f(x) = -3x^5 + 5x^3$

3.5 A Summary of Curve Sketching

Guidelines for Analyzing the Graph of a Function

1. Determine the domain and range of the function.
2. Determine the intercepts, asymptotes, and symmetry of the graph.
3. Locate the x-values for which $f'(x)$ and $f''(x)$ either are zero or do not exist. Use the results to determine relative extrema and points of inflection.

Multiple Choice What are the equations of the asymptotes of the function

$$f(x) = \frac{x^3 - 1}{x^2 - x}?$$

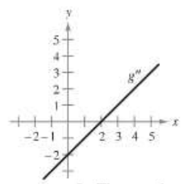
(A) $x = 0, x = 1$

(B) $x = 0, x = 1, y = x + 1$

(C) $x = 0, y = x + 1$

(D) $x = 0, x = 1, y = x$

. **Multiple Choice** The figure shows the graph of the second derivative of a function g. Which of the following statements are true?



I. The graph of g is concave upward on the interval $(2, \infty)$.

II. g' is decreasing on the interval $(-\infty, 2)$.

III. The graph of g has a point of inflection at $x = 2$.

(A) I and II only (B) I and III only

(C) III only (D) I, II, and III

3.5 A Summary of Curve Sketching

Example 1: Sketch the graph of a rational function $f(x) = \dfrac{x^2}{2(x^2 - 16)}$

3.5 A Summary of Curve Sketching

Example 2: Sketch the graph of a logistic function $f(x) = \dfrac{3}{1+e^{2x}}$

3.5 A Summary of Curve Sketching

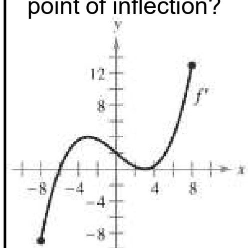
Example 5: The graph of the first derivative of a function f on the interval $[-8,8]$ is shown. Use the graph to answer each question.

a. On what interval(s) is f increasing?

b. On what interval(s) is the graph of f concave downward?

c. At what x-value(s) does f have relative extrema?

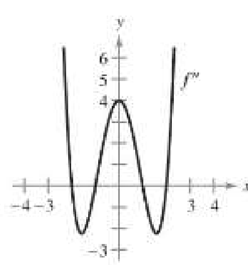
d. At what x-value(s) does the graph of f have a point of inflection?



Example 6: The graph of the second derivative of a function f is shown. Use the graph to answer each question.

a. On what intervals is the graph of f concave downward?

b. On what intervals is f' increasing?



Free Response The derivative of a function f is $f'(x) = -12(x-2)^2(x-4)$.

(a) Does f have any relative extrema? If so, where do they occur?

(b) On what interval(s) is f decreasing?

(c) On what interval(s) is the graph of f concave downward?

(d) How many point(s) of inflection does the graph of f have? Where do they occur?