Definition of Extrema

Let f be defined on an interval I containing c.

1. $f(c)$ is the **minimum of f on I** when $f(c) \le f(x)$ for all x in I.
2. $f(c)$ is the **maximum of f on I** when $f(c) \ge f(x)$ for all x in I.

The minimum and maximum of a function on an interval are the **extreme values,** or **extrema** (the singular form of extrema is extremum), of the function on the interval. The minimum and maximum of a function on an interval are also called the **absolute minimum** and **absolute maximum,** or the **global minimum** and **global maximum,** on the interval. Extrema can occur at interior points or endpoints of an interval. (See Figure 3.1.) Extrema that occur at the endpoints are called **endpoint extrema.**

THEOREM 3.1 The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f has both a minimum and a maximum on the interval.

Exploration

Finding Minimum and Maximum Values The Extreme Value Theorem (like the Intermediate Value Theorem) is an *existence theorem* because it tells of the existence of minimum and maximum values but does not show how to find these values. Use the *minimum* and *maximum* features of a graphing utility to find the extrema of each function. In each case, do you think the x-values are exact or approximate? Explain your reasoning.

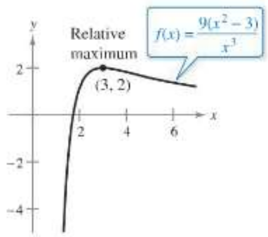
a. $f(x) = x^2 - 4x + 5$ on the closed interval $[-1, 3]$

b. $f(x) = x^3 - 2x^2 - 3x - 2$ on the closed interval $[-1, 3]$

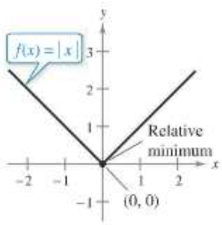
3.1 Extrema on an Interval

Example 1: Find the value of the derivative at each relative extremum shown.

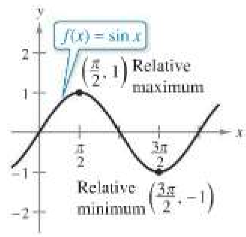
a. $f(x) = \frac{9(x^2-3)}{x^3}$ at $(3,2)$



b. $f(x) = |x|$ at $(0,0)$



c. $f(x) = \sin x$ at $\left(\frac{\pi}{2}, 1\right)$ and $\left(\frac{3\pi}{2}, -1\right)$



3.1 Extrema on an Interval

Definition of Relative Extrema

1. If there is an open interval containing c on which $f(c)$ is a maximum, then $f(c)$ is called a **relative maximum** of f, or you can say that f has a **relative maximum at** $(c, f(c))$.
2. If there is an open interval containing c on which $f(c)$ is a minimum, then $f(c)$ is called a **relative minimum** of f, or you can say that f has a **relative minimum at** $(c, f(c))$.

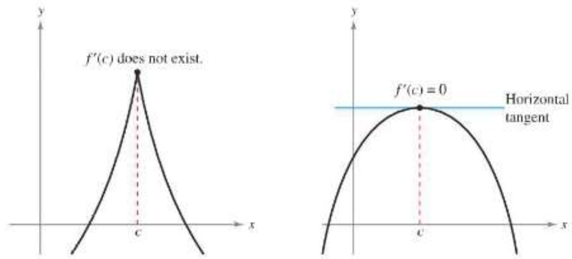
The plural of relative maximum is relative maxima, and the plural of relative minimum is relative minima. Relative maximum and relative minimum are sometimes called **local maximum** and **local minimum,** respectively.

Definition of a Critical Number

Let f be defined at c. If $f'(c) = 0$ or if f is not differentiable at c, then c is a **critical number** of f.

Insight

On the AP® Exam, critical numbers are sometimes referred to as "critical values."



c is a critical number of f.

Figure 3.4

Technology

Use a graphing utility to examine the graphs of the following four functions. Only one of the functions has critical numbers. Which is it?

$f(x) = e^x$ $\quad f(x) = \ln x$

$f(x) = \sin x$ $\quad f(x) = \tan x$

THEOREM 3.2 Relative Extrema Occur Only at Critical Numbers

If f has a relative minimum or relative maximum at $x = c$, then c is a critical number of f.

Guidelines for Finding Extrema on a Closed Interval

To find the extrema of a continuous function f on a closed interval $[a, b]$, use these steps.

1. Find the critical numbers of f in (a, b).
2. Evaluate f at each critical number in (a, b).
3. Evaluate f at each endpoint of $[a, b]$.
4. The least of these values is the minimum. The greatest is the maximum.

Insight

When an AP® Exam question asks for an absolute extremum of a function on a closed interval, be sure to evaluate the function at the endpoints of the interval.

Algebra Review

For help on the algebra in Example 2, see Example 1(a) in the *Chapter 3 Algebra Review* on page A40.

3.1 Extrema on an Interval

Example 2: Find the extrema of the following functions on the indicated intervals.

a. $f(x) = 3x^4 - 4x^3$ on the interval $[-1,2]$

b. $f(x) = 2x - 3x^{2/3}$ on the interval $[-1,3]$

c. $f(x) = 2\sin x - \cos 2x$ on the interval $[0,2\pi]$

3.2 Rolle's Theorem and the Mean Value Theorem

THEOREM 3.3 Rolle's Theorem

Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) such that $f'(c) = 0$.

Example 1: Find the two x-intercepts of $f(x) = x^2 - 3x + 2$ and show that $f'(x) = 0$ at some point between the two x-intercepts.

3.2 Rolle's Theorem and the Mean Value Theorem

Example 2: Let $f(x) = x^4 - 2x^2$. Find all values of c in the interval (-2,2) such that $f'(x) = 0$.

3.2 Rolle's Theorem and the Mean Value Theorem

THEOREM 3.4 The Mean Value Theorem

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Remark

The "mean" in the Mean Value Theorem refers to the mean (or average) rate of change of f on the interval $[a, b]$.

Example 3: For $f(x) = 5 - \left(\frac{4}{x}\right)$, find all values of c in the open interval (1,4) such that $f'(x) = \frac{f(4)-f(1)}{4-1}$.

Example 4: Two stationary patrol cars equipped with radar are 5 miles apart on a highway, as shown in the picture. As a truck passes the first patrol car, its speed is clocked at 55 miles per hour. Four minutes later, when the truck passes the second patrol car, its speed is clocked at 50 miles per hour. Prove that the truck must have exceeded the speed limit (of 55 miles per hour) at some time during the 4 minutes.