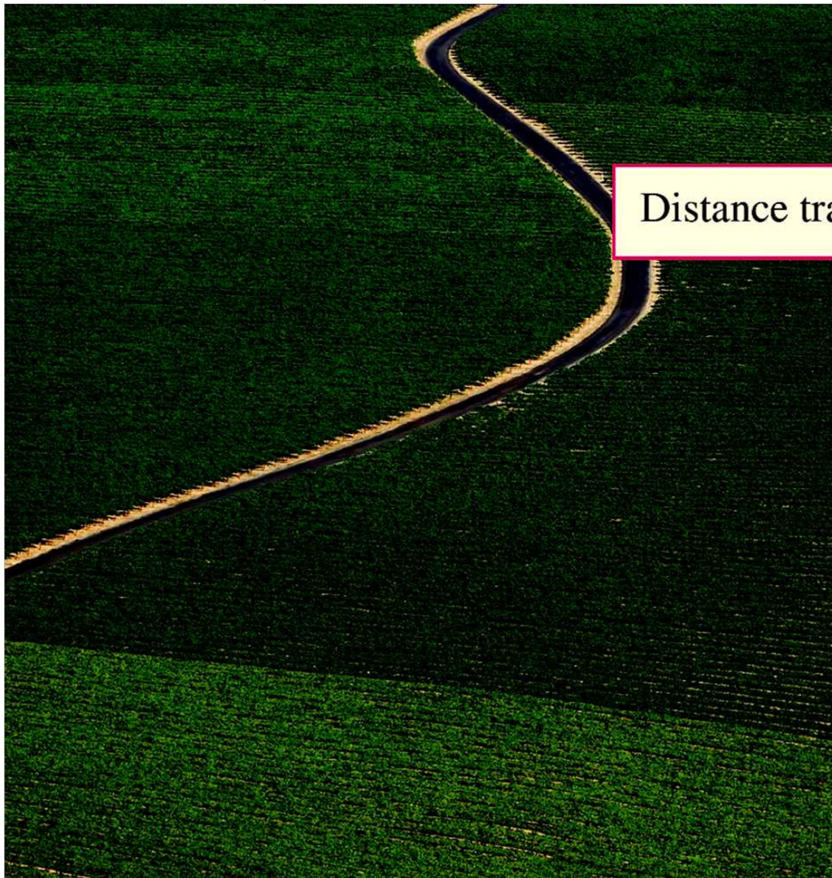
Calculus Notes 5.1: Approximating and Computing Area.

How would you find the area of the field with that road as a boundary?



$$\text{Distance traveled} = \overbrace{\text{velocity} \times \text{time elapsed}}^{v\Delta t}$$

$$\text{Distance traveled} = \text{area under the graph of velocity over } [t_1, t_2]$$

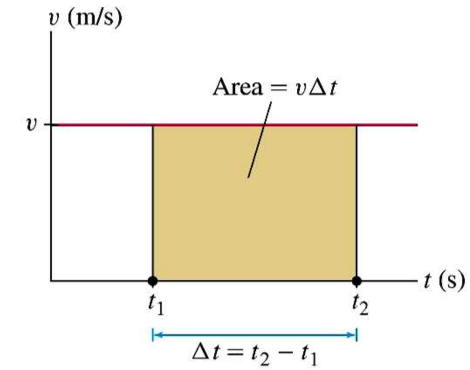


FIGURE 1 The rectangle has area $v\Delta t$, which is equal to the distance traveled.

Say a road runner runs for 2 seconds at 5 m/s, then another second at 15 m/s, 3 more seconds at 10 m/s, and last 2 more seconds at 5 m/s. How far does it travel total?

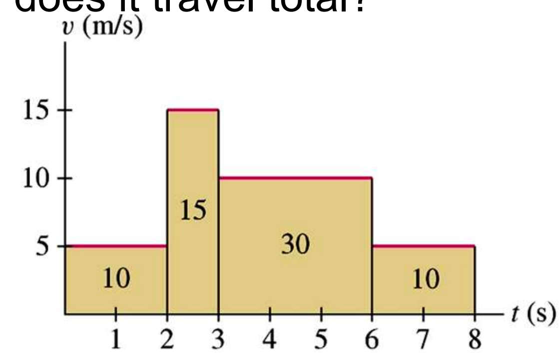


FIGURE 2 Distance traveled equals the sum of the areas of the rectangles.

Calculus Notes 5.1: Approximating and Computing Area.

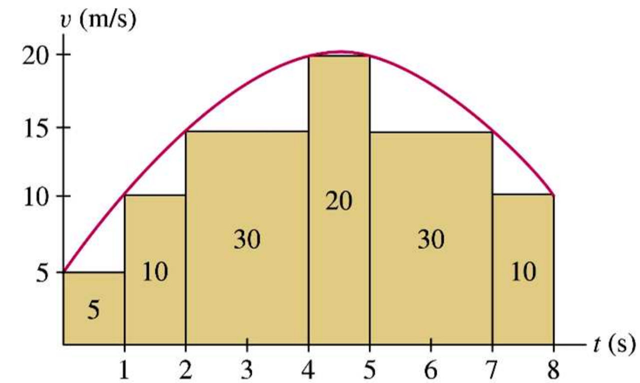
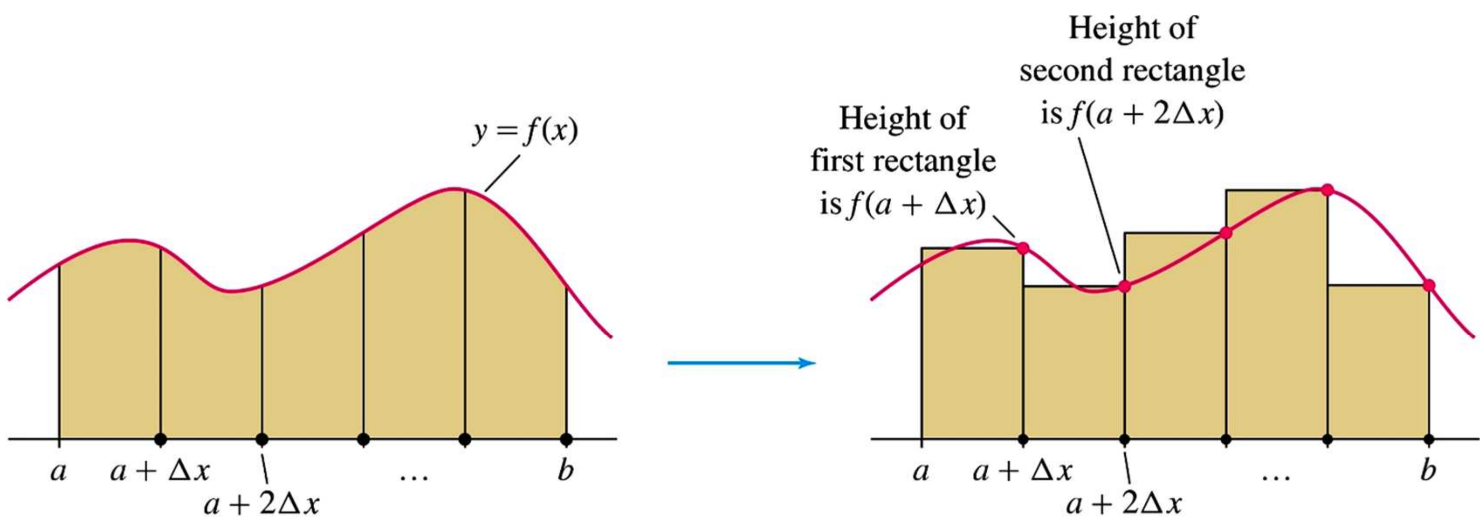


FIGURE 3 Distance traveled is equal to the area under the graph. It is *approximated* by the sum of the areas of the rectangles.



(A) Divide $[a, b]$ into N subintervals, each of width Δx

(B) Construct right-endpoint rectangles

FIGURE 4

Calculus Notes 5.1: Approximating and Computing Area.

$$R_N = \Delta x \left(f(a+\Delta x) + f(a+2\Delta x) + \cdots + f(a+N\Delta x) \right)$$

Example 1: Calculate R_4 and R_6 for $f(x) = x^2$ on the interval $[1,3]$.
Then calculate R_5 , L_5 , and M_5 .

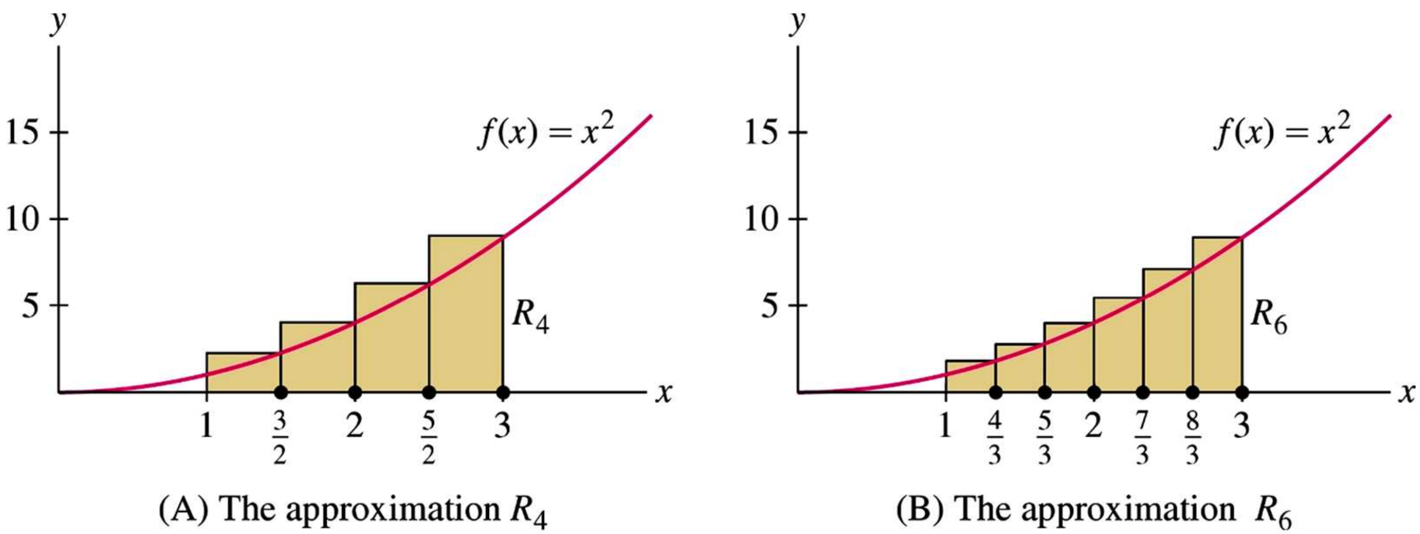


FIGURE 5

Which one is an overestimate? Underestimate?

Calculus Notes 5.1: Approximating and Computing Area.

Linearity of Summations

- $$\sum_{j=m}^{n}(a_j + b_j) = \sum_{j=m}^{n} a_j + \sum_{j=m}^{n} b_j$$
- $$\sum_{j=m}^{n} Ca_j = C\sum_{j=m}^{n} a_j \qquad (C \text{ any constant})$$
- $$\sum_{j=1}^{n} k = nk \qquad (k \text{ any constant and } n \geq 1)$$

$$R_N = \Delta x \sum_{j=1}^{N} f(a + j\Delta x)$$

$$L_N = \Delta x \sum_{j=0}^{N-1} f(a + j\Delta x)$$

$$M_N = \Delta x \sum_{j=1}^{N} f\left(a + \left(j - \frac{1}{2}\right)\Delta x\right)$$

$$\Delta x = \left(\frac{b-a}{N}\right)$$

Example 2: Calculate R_8, L_8, and M_8 for $f(x) = x^{-1}$ on the interval $[2,4]$.

Calculus Notes 5.1: Approximating and Computing Area.

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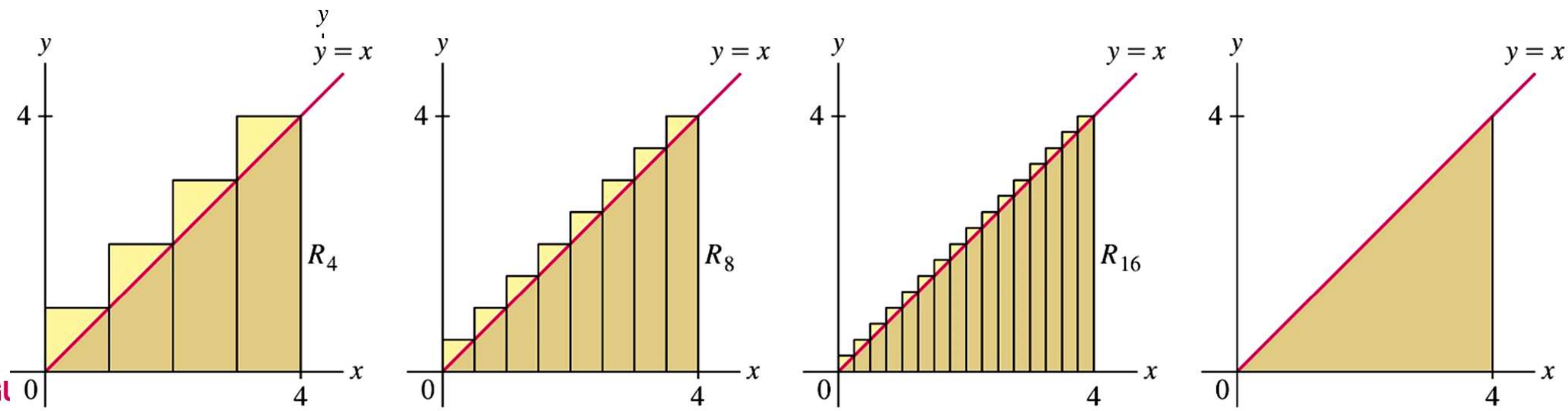


FIGURE 11 The right-endpoint approximations approach the area of the triangle.

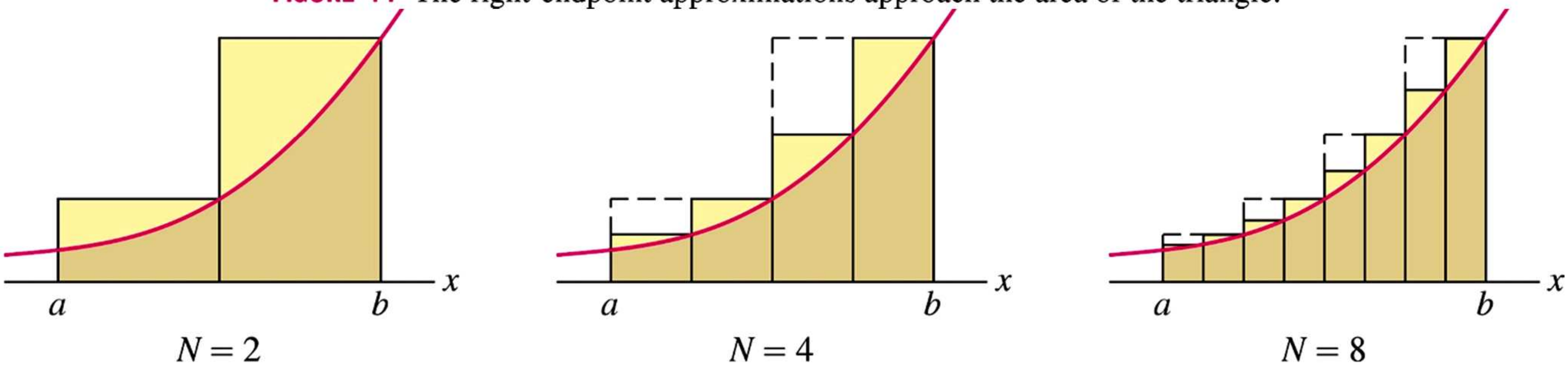


FIGURE 10 The error decreases as we use more rectangles.

Calculus Notes 5.1: Approximating and Computing Area.

Linearity of Summations

- $$\sum_{j=m}^{n}(a_j + b_j) = \sum_{j=m}^{n} a_j + \sum_{j=m}^{n} b_j$$
- $$\sum_{j=m}^{n} Ca_j = C\sum_{j=m}^{n} a_j \qquad (C \text{ any constant})$$
- $$\sum_{j=1}^{n} k = nk \qquad (k \text{ any constant and } n \geq 1)$$

$$R_N = \Delta x \sum_{j=1}^{N} f(a + j\Delta x)$$

$$L_N = \Delta x \sum_{j=0}^{N-1} f(a + j\Delta x)$$

$$M_N = \Delta x \sum_{j=1}^{N} f\left(a + \left(j - \frac{1}{2}\right)\Delta x\right)$$

$$\Delta x = \left(\frac{b-a}{N}\right)$$

Calculus Notes 5.1: Approximating and Computing Area.

THEOREM 1 If $f(x)$ is continuous on $[a, b]$, then the endpoint and midpoint approximations approach one and the same limit as $N \to \infty$. In other words, there is a value L such that

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty} L_N = \lim_{N\to\infty} M_N = L$$

If $f(x) \geq 0$, we define the area under the graph over $[a, b]$ to be L.

Power Sums

$$\sum_{j=1}^{N} j = 1 + 2 + \cdots + N = \frac{N(N+1)}{2} = \frac{N^2}{2} + \frac{N}{2} \qquad \boxed{3}$$

$$\sum_{j=1}^{N} j^2 = 1^2 + 2^2 + \cdots + N^2 = \frac{N(N+1)(2N+1)}{6} = \frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6} \qquad \boxed{4}$$

$$\sum_{j=1}^{N} j^3 = 1^3 + 2^3 + \cdots + N^3 = \frac{N^2(N+1)^2}{4} = \frac{N^4}{4} + \frac{N^3}{2} + \frac{N^2}{4} \qquad \boxed{5}$$

Calculus Notes 5.1: Approximating and Computing Area.

Example 1: Use linearity and formulas [3]—[5] to rewrite and evaluate the sums.

a. $\displaystyle\sum_{n=51}^{150} n^2$

b. $\displaystyle\sum_{j=0}^{50} j(j-1)$

Calculus Notes 5.1: Approximating and Computing Area.

Example 2: Calculate the limit for the given function and interval. Verify your answer by using geometry.

$a.\quad \lim_{N\to\infty} R_N \quad f(x)=9x \quad [0,2]$

$b.\quad \lim_{N\to\infty} L_N \quad f(x)=\frac{1}{2}x+2 \quad [0,4]$

Calculus Notes 5.1: Approximating and Computing Area.

Example 3: Find a formula for R_N and compute the area under the graph as a limit.

$a.\quad f(x)=\frac{1}{2}x+2\ [0,4]$

$b.\quad f(x)=x^2\ [-1,5]$

Calculus Notes 5.2: The Definite Integral

From 5.1:

THEOREM 1 If $f(x)$ is continuous on $[a, b]$, then the endpoint and midpoint approximations approach one and the same limit as $N \to \infty$. In other words, there is a value L such that

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty} L_N = \lim_{N\to\infty} M_N = L$$

If $f(x) \geq 0$, we define the area under the graph over $[a, b]$ to be L.

When you add up the rectangles, do they all have to be the same width?

For Riemann sum approximations, we relax the requirements that the rectangles have to have equal width.

Calculus Notes 5.2: The Definite Integral

$$R(f, P, C) = \sum_{i=1}^{N} f(c_i)\Delta x_i = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \cdots + f(c_N)\Delta x_N$$

$$\Delta x_i = x_i - x_{i-1}$$

Calculus Notes 5.2: The Definite Integral

$$R(f, P, C) = \sum_{i=1}^{N} f(c_i)\Delta x_i = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \cdots + f(c_N)\Delta x_N$$

Example 1: Calculate $R(f,P,C)$, where $f(x) = 7 + 2\cos x - 3x$ on $[0,4]$

$$P: x_0 = 0 < x_1 = 1 < x_2 = 1.7 < x_3 = 3.1 < x_4 = 4$$

$$C: \{0.3, 1.1, 2.5, 3.4\}$$

What is the norm $\|P\|$?

Calculus Notes 5.2: The Definite Integral

DEFINITION Definite Integral The definite integral of $f(x)$ over $[a, b]$, denoted by the integral sign, is the limit of Riemann sums:

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} R(f, P, C) = \lim_{\|P\|\to 0} \sum_{i=1}^{N} f(c_i)\Delta x_i$$

When this limit exists, we say that $f(x)$ is integrable over $[a, b]$.

THEOREM 1 If $f(x)$ is continuous on $[a, b]$, or if $f(x)$ is continuous with at most finitely many jump discontinuities, then $f(x)$ is integrable over $[a, b]$.

Calculus Notes 5.2: The Definite Integral

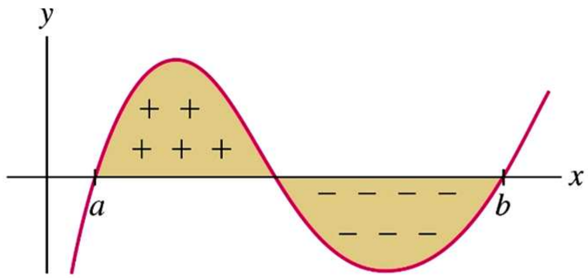


FIGURE 4 Signed area is the area above the x-axis minus the area below the x-axis.

Signed area of a region = (area above x-axis) − (area below x-axis)

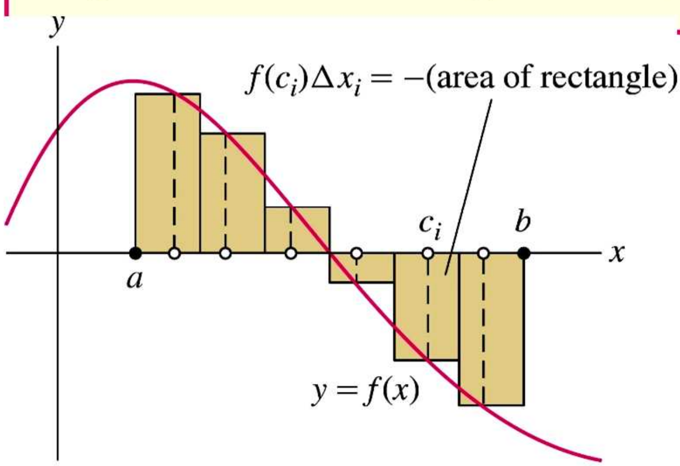


FIGURE 5

$$\int_a^b f(x)\,dx = \text{signed area of region between the graph and } x\text{-axis over } [a, b]$$

Calculus Notes 5.2: The Definite Integral

THEOREM 2 Integral of a Constant For any constant C,

$$\int_a^b C\,dx = C(b-a)$$

THEOREM 3 Linearity of the Definite Integral If f and g are integrable over $[a, b]$, then $f + g$ and Cf are integrable (for any constant C), and

- $\int_a^b \big(f(x) + g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
- $\int_a^b Cf(x)\,dx = C\int_a^b f(x)\,dx$

$$\int_0^b x^2\,dx = \frac{b^3}{3}$$

DEFINITION Reversing the Limits of Integration For $a < b$, we set

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

Calculus Notes 5.2: The Definite Integral

Example 2: Calculate

$$\int_0^5 (3 - x)\,dx \qquad \text{and} \qquad \int_0^5 |3 - x|\,dx$$

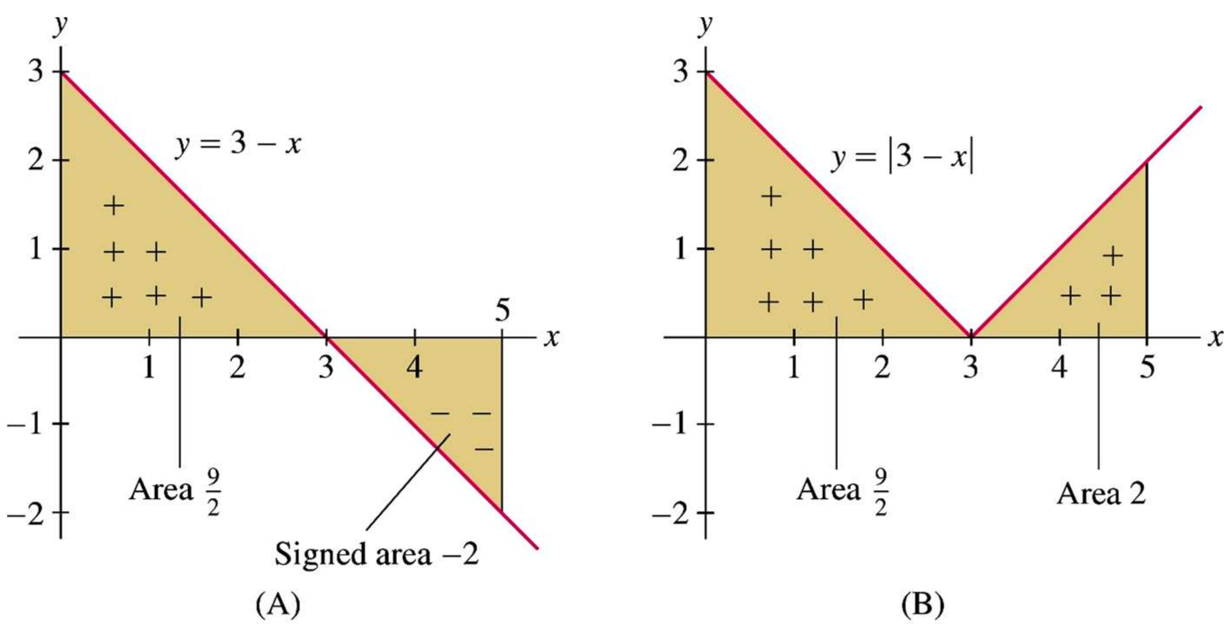


FIGURE 6

Calculus Notes 5.2: The Definite Integral

$$\int_a^a f(x)\,dx = 0$$

$$\int_0^b x\,dx = \frac{1}{2}b^2$$

THEOREM 4 Additivity for Adjacent Intervals Let $a \le b \le c$, and assume that $f(x)$ is integrable. Then

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

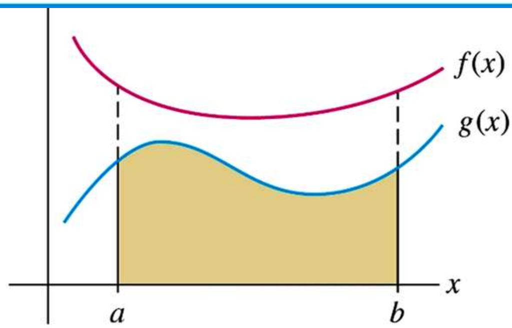
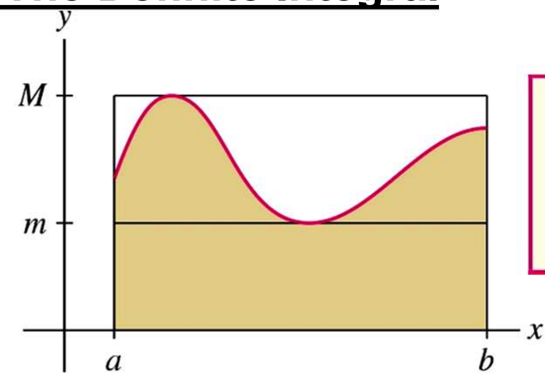


FIGURE 10 The integral of $f(x)$ is larger than the integral of $g(x)$.

THEOREM 5 Comparison Theorem If f and g are integrable and $g(x) \le f(x)$ for x in $[a, b]$, then

$$\int_a^b g(x)\,dx \le \int_a^b f(x)\,dx$$

Calculus Notes 5.2: The Definite Integral



$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

FIGURE 12 The integral $\int_a^b f(x)\,dx$ lies between the areas of the rectangles of heights m and M.

Example 3: Calculate $\int_4^7 \left(x + x^2\right) dx$

Calculus Notes 5.3: The Fundamental Theorem of Calculus, Part I

Recall from 5.2 Reading Example 5: $\int_4^7 x^2 dx$

$$\int_0^4 x^2 dx + \int_4^7 x^2 dx = \int_0^7 x^2 dx \Rightarrow \int_4^7 x^2 dx = \int_0^7 x^2 dx - \int_0^4 x^2 dx$$

$$\boxed{\int_0^b x^2\,dx = \frac{b^3}{3}} \Rightarrow \frac{7^3}{3} - \frac{4^3}{3} = \frac{1}{3}\left(7^3 - 4^3\right) = \frac{1}{3}(279) = 93$$

Note that $F(x) = \frac{1}{3}x^3$ is an antiderivative of x^2

THEOREM 1 The Fundamental Theorem of Calculus, Part I Assume that $f(x)$ is continuous on $[a, b]$. If $F(x)$ is an antiderivative of $f(x)$ on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

So we can re-write it as:

$$\int_4^7 x^2 dx = F(7) - F(4)$$

Calculus Notes 5.2: The Definite Integral

Example 1: evaluate the integral using FTC I:

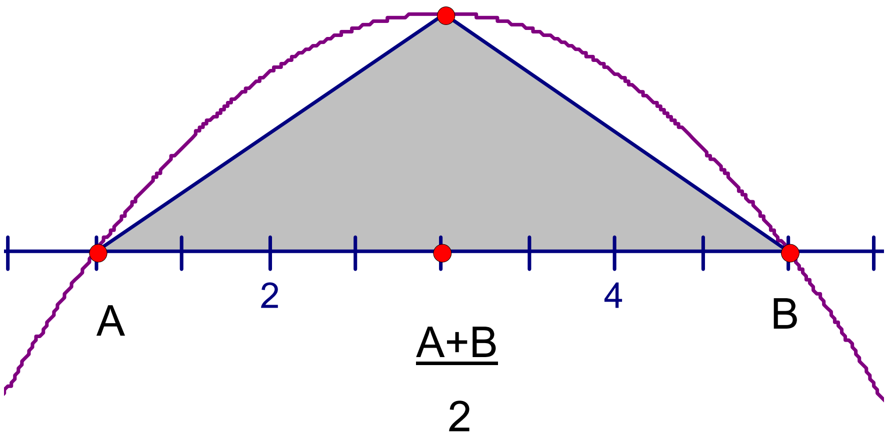
$$a.\ \int_0^9 2\,dx$$

$$b.\ \int_{-1}^{1}\left(5u^4 + u^2 - u\right)du$$

$$c.\ \int_{\pi/4}^{3\pi/4} \sin\theta\, d\theta$$

Calculus Notes 5.2: The Definite Integral

Example 2: Show that the area of the shaded parabolic arch in figure 8 is equal to four-thirds the area of the triangle shown.



$$y = (x - a)(b - x)$$

Calculus Notes 5.4: The Fundamental Theorem of Calculus, Part II

From 5.3

THEOREM 1 The Fundamental Theorem of Calculus, Part I Assume that $f(x)$ is continuous on $[a, b]$. If $F(x)$ is an antiderivative of $f(x)$ on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

From 5.4

THEOREM 1 Fundamental Theorem of Calculus, Part II Assume that $f(x)$ is continuous on an open interval I and let $a \in I$. Then the area function

$$A(x) = \int_a^x f(t)\,dt$$

is an antiderivative of $f(x)$ on I; that is, $A'(x) = f(x)$. Equivalently,

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Furthermore, $A(x)$ satisfies the initial condition $A(a) = 0$.

Calculus Notes 5.4: The Fundamental Theorem of Calculus, Part II

Proof of FTC II:

THEOREM 1 Fundamental Theorem of Calculus, Part II Assume that $f(x)$ is continuous on an open interval I and let $a \in I$. Then the area function

$$A(x) = \int_a^x f(t)\,dt$$

is an antiderivative of $f(x)$ on I; that is, $A'(x) = f(x)$. Equivalently,

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Furthermore, $A(x)$ satisfies the initial condition $A(a) = 0$.

First, use the additive property of the definite integral to write the change in A(x) over [x,x+h].

$$A(x+h) - A(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \int_x^{x+h} f(t)\,dt$$

In other words, A(x+h)-A(x) is equal to the area of the thin section between the graph and the x-axis from x to x+h.

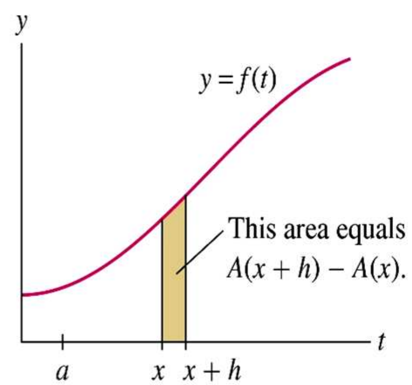


FIGURE 3 The area of the thin sliver equals $A(x + h) - A(x)$.

Calculus Notes 5.4: The Fundamental Theorem of Calculus, Part II

Proof of FTC II continued:

THEOREM 1 Fundamental Theorem of Calculus, Part II Assume that $f(x)$ is continuous on an open interval I and let $a \in I$. Then the area function

$$A(x) = \int_a^x f(t)\,dt$$

is an antiderivative of $f(x)$ on I; that is, $A'(x) = f(x)$. Equivalently,

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Furthermore, $A(x)$ satisfies the initial condition $A(a) = 0$.

To simplify the rest of the proof, we assume that f(x) is increasing. Then, if h>0, this thin section lies between the two rectangles of heights f(x) and f(x+h).

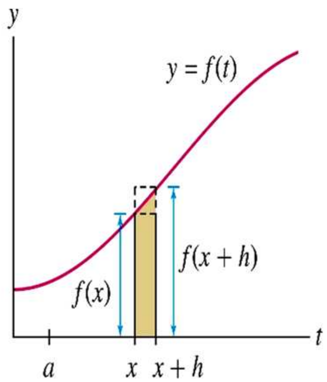


FIGURE 4 The shaded sliver lies between the rectangles of heights $f(x)$ and $f(x+h)$.

So we have:

$$hf(x) \leq A(x+h) - A(x) \leq hf(x+h)$$

Area of small rectangle | Area of section | Area of big rectangle

Now divide by h to squeeze the difference quotient between f(x) and f(x+h)

$$f(x) \leq \frac{A(x+h) - A(x)}{h} \leq f(x+h)$$

We have $\lim_{h\to 0+} f(x+h) = f(x)$ because f(x) is continuous, and $\lim_{h\to 0-} f(x) = f(x)$

So the Squeeze Theorem gives us: $$\lim_{h\to 0+} \frac{A(x+h) - A(x)}{h} = f(x)$$

A similar argument shows that for h<0: $$\lim_{h\to 0-} \frac{A(x+h) - A(x)}{h} = f(x)$$

Calculus Notes 5.4: The Fundamental Theorem of Calculus, Part II

Proof of FTC II continued:

THEOREM 1 Fundamental Theorem of Calculus, Part II Assume that $f(x)$ is continuous on an open interval I and let $a \in I$. Then the area function

$$A(x) = \int_a^x f(t)\,dt$$

is an antiderivative of $f(x)$ on I; that is, $A'(x) = f(x)$. Equivalently,

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Furthermore, $A(x)$ satisfies the initial condition $A(a) = 0$.

Equation 1

$$\lim_{h\to 0+} \frac{A(x+h) - A(x)}{h} = f(x)$$

Equation 2

$$\lim_{h\to 0-} \frac{A(x+h) - A(x)}{h} = f(x)$$

Equation 1 and Equation 2 show that $A'(x)$ exists and $A'(x) = f(x)$.

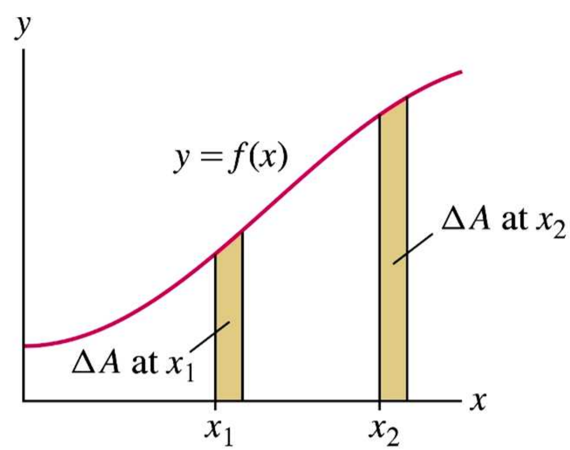


FIGURE 6 The change in area ΔA for a given Δx is larger when $f(x)$ is larger.

Calculus Notes 5.4: The Fundamental Theorem of Calculus, Part II

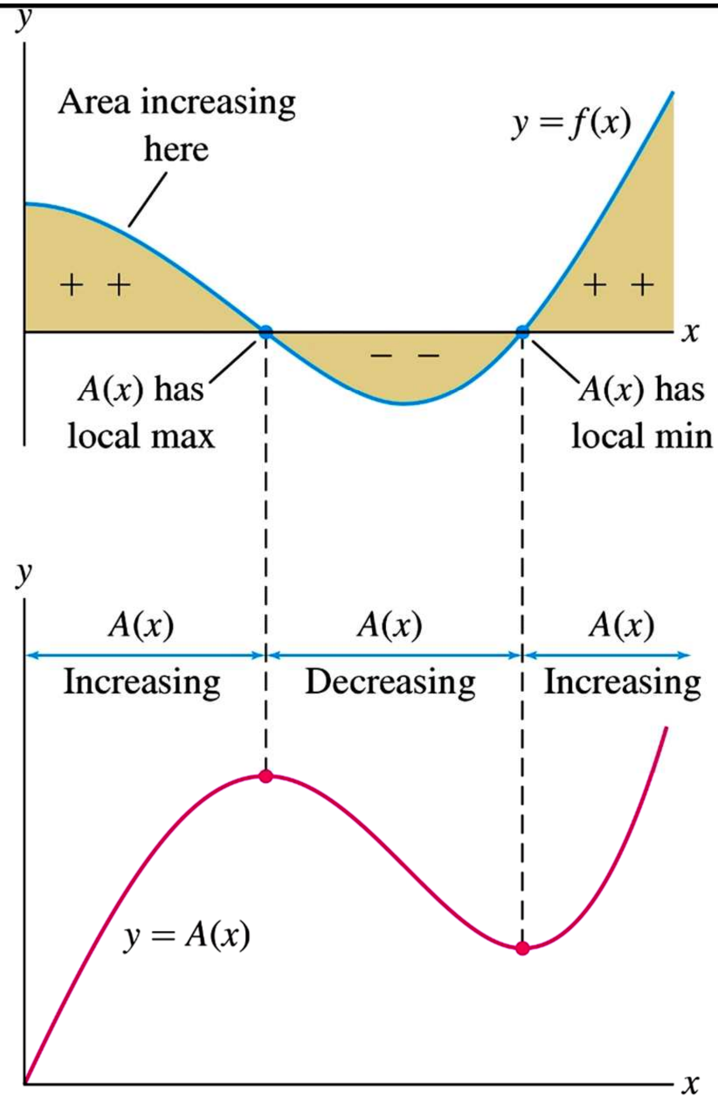


FIGURE 7 The sign of $f(x)$ determines the increasing/decreasing behavior of $A(x)$.

Calculus Notes 5.4: The Fundamental Theorem of Calculus, Part II

Example 1: Find $F(1)$, $F'(0)$, and $F'\left(\frac{\pi}{4}\right)$, where $F(x) = \int_1^x \tan t\,dt$

Calculus Notes 5.4: The Fundamental Theorem of Calculus, Part II

Example 2: Find formulas for the functions represented by the integrals.

a. $\int_2^x (12t^2 - 8t)\,dt$

b. $\int_x^0 e^{-t}\,dt$

c. $\int_2^{\sqrt{x}} \frac{dt}{t}$

Calculus Notes 5.4: The Fundamental Theorem of Calculus, Part II

Example 3: Calculate the derivative.

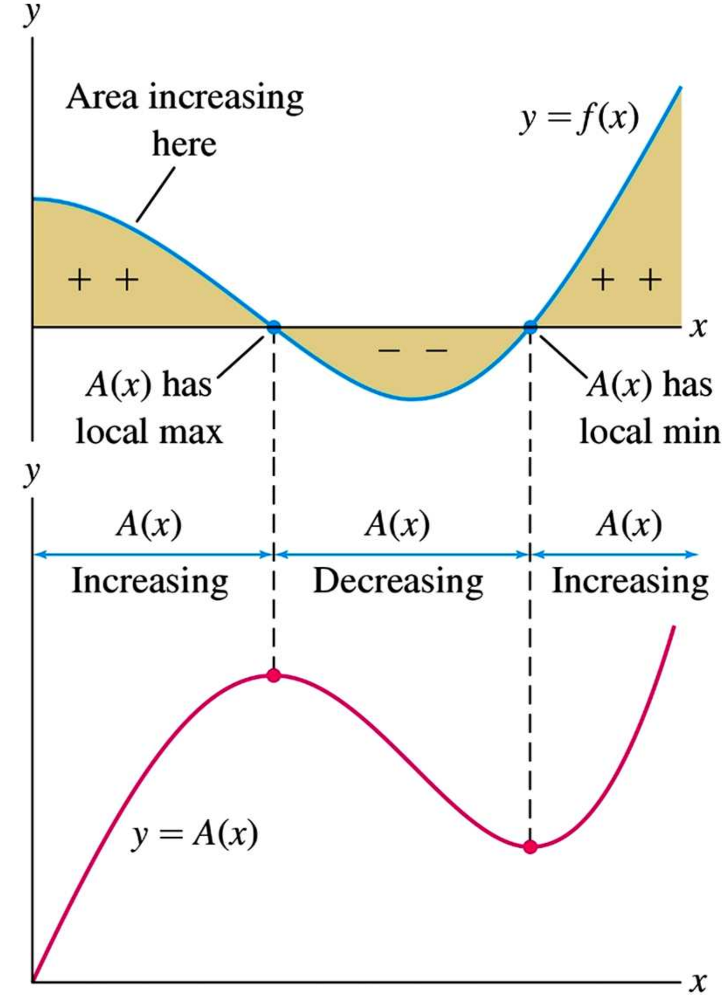
a. $\frac{d}{d\theta}\int_1^0 (\cot u)\,du$

b. $\frac{d}{dx}\int_0^{x^2} \frac{t}{t+1}\,dt$

Calculus Notes 5.4: The Fundamental Theorem of Calculus, Part II

Example 4: Explain why the following statements are true. Assume f(x) is differentiable.

(a) If c is an inflection point of A(x), then $f'(c) = 0$



(b) A(x) is concave up if f(x) is increasing.

(c) A(x) is concave down if f(x) is decreasing.

Calculus Notes 5.5 & 5.6: Net or Total Change as the Integral of a Rate and Substitution Method.

Position function: $s(t)$

Velocity function: $s'(t) = v(t)$

Acceleration function: $s''(t) = v'(t) = a(t)$

THEOREM 1 Net Change as the Integral of a Rate The net change in $s(t)$ over an interval $[t_1, t_2]$ is given by the integral

$$\underbrace{\int_{t_1}^{t_2} s'(t)\,dt}_{\text{Integral of the rate of change}} = \underbrace{s(t_2) - s(t_1)}_{\text{Net change over } [t_1, t_2]}$$

THEOREM 2 The Integral of Velocity For an object in linear motion with velocity $v(t)$, then

$$\text{Displacement during } [t_1, t_2] = \int_{t_1}^{t_2} v(t)\,dt$$

$$\text{Distance traveled during } [t_1, t_2] = \int_{t_1}^{t_2} |v(t)|\,dt$$

Example 1: Find the total displacement and total distance traveled in the interval [0,6] by a particle moving in a straight line with velocity

$$v(t) = 2t - 3 \ m/s$$

$$\int_0^6 v(t)dt = \int_0^6 2t - 3dt$$

$$= \left(t^2 - 3t\right)_1^6$$

$$= (6)^2 - 3(6) - \left((0)^2 - 3(0)\right)$$

$$= 36 - 18 + 0 = 18m$$

$$\int_0^6 |v(t)|dt = \int_0^6 |2t - 3|dt = \int_0^{1.5} 3 - 2tdt + \int_{1.5}^6 2t - 3dt$$

$$0 = 2t - 3 \ m/s \rightarrow t = \frac{3}{2}$$

$$v(t) < 0 \text{ for } t < \frac{3}{2}$$

$$v(t) > 0 \text{ for } t > \frac{3}{2}$$

$$= \int_{1.5}^6 2t - 3dt - \int_0^{1.5} 2t - 3dt$$

$$= \left(t^2 - 3t\right)_{1.5}^6 - \left(t^2 - 3t\right)_0^{1.5}$$

$$= (18 - (-2.25)) - ((-2.25) - 0)$$

$$= 22.5$$

Calculus Notes 5.5 & 5.6: Net or Total Change as the Integral of a Rate and Substitution Method.

Example 2: The number of cars per hour passing an observation point along a highway is called the traffic flow rate q(t) (in cars per hour).

a. Which quantity is represented by the integral $\int_{t_1}^{t_2} q(t)\,dt$

The integral $\int_{t_1}^{t_2} q(t)\,dt$ represents the total number of cars that passed the observation point during the time interval $[t_1, t_2]$

b. The flow rate is recorded at 15-minute intervals between 7:00 and 9:00 AM. Estimate the number of cars using the highway during this 2-hour period.

t	7:00	7:15	7:30	7:45	8:00	8:15	8:30	8:45	9:00
$q(t)$	1044	1297	1478	1844	1451	1378	1155	802	542

Calculus Notes 5.5 & 5.6: Net or Total Change as the Integral of a Rate and Substitution Method.

Find the integral of the following:

a. $\int 2x\left(x^2+9\right)^4 dx$

b. $\int 2x\cos\left(x^2\right)$

c. $\int \sqrt{1+2x}\,dx$

Calculus Notes 5.5 & 5.6: Net or Total Change as the Integral of a Rate and Substitution Method.

Find the integral of the following:

d. $\int \frac{x}{\sqrt{x^2+9}}\,dx$

e. $\int \cos(x^2)\,dx$

f. $\int \sqrt{1+2x^2}\,dx$

Calculus Notes 5.5 & 5.6: Net or Total Change as the Integral of a Rate and Substitution Method.

THEOREM 1 The Substitution Method If $F'(x) = f(x)$, then

$$\int f(u(x))u'(x)\,dx = F(u(x)) + C$$

$$du = \frac{du}{dx}dx$$

so

$$du = u'(x)\,dx$$

so

$$\int \underbrace{f(u(x))}_{f(u)} \underbrace{u'(x)\,dx}_{du} = \int f(u)\,du$$

This equation is called the **Change of Variables Formula.**

Calculus Notes 5.5 & 5.6: Net or Total Change as the Integral of a Rate and Substitution Method.

Example 3a: $\int \frac{x+3}{x^2+6x+1}\,dx$

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Change of Variables Formula for Definite Integrals

$$\int_a^b f(u(x))u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du$$

Example 3b: $\int_0^2 \frac{x+3}{x^2+6x+1}\,dx$

Calculus Notes 5.7 & 5.8: Further Transcendental Functions & Exponential Growth and Decay

Calculate $\int_1^x \frac{dt}{t} = \ln x$ So $\int_a^b \frac{dt}{t} = \ln t\Big|_a^b = \ln b - \ln a = \ln\left|\frac{b}{a}\right|$

$$\ln x = \int_1^x \frac{dt}{t} \qquad \text{for } x > 0$$

$$\int b^x\,dx = \frac{b^x}{\ln b} + C$$

$$\int e^x dx = e^x$$

Inverse Trigonometric Functions

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}, \qquad \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + C$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{x^2+1}, \qquad \int \frac{dx}{x^2+1} = \tan^{-1}x + C$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}, \qquad \int \frac{dx}{|x|\sqrt{x^2-1}} = \sec^{-1}x + C$$

Calculus Notes 5.7 & 5.8: Further Transcendental Functions & Exponential Growth and Decay

Example 1: Evaluate the definite integral $\int_{2}^{12} \frac{dt}{3t+4}$

Calculus Notes 5.7 & 5.8: Further Transcendental Functions & Exponential Growth and Decay

Example 2: Evaluate the definite integral $\displaystyle\int_{-1/5}^{1/5} \frac{dx}{\sqrt{4-25x^2}}$

Calculus Notes 5.7 & 5.8: Further Transcendental Functions & Exponential Growth and Decay

Example 3: Evaluate the indefinite integral $\displaystyle\int \frac{x\,dx}{x^4+1}$

Calculus Notes 5.7 & 5.8: Further Transcendental Functions & Exponential Growth and Decay

Example 4: Evaluate the definite integral $\int_0^3 3^{-x}\,dx$

Calculus Notes 5.7 & 5.8: Further Transcendental Functions & Exponential Growth and Decay

$$P(t) = P_0 e^{kt}$$

THEOREM 1 If $y(t)$ is a differentiable function satisfying the differential equation

$$y' = ky$$

then $y(t) = P_0 e^{kt}$, where P_0 is the initial value $P_0 = y(0)$.

Doubling Time If $P(t) = P_0 e^{kt}$ with $k > 0$, then the doubling time of P is

$$\text{Doubling time} = \frac{\ln 2}{k}$$

$$\text{Half-life} = \frac{\ln 2}{k}$$

Calculus Notes 5.7 & 5.8: Further Transcendental Functions & Exponential Growth and Decay

Compound Interest If P_0 dollars are deposited into an account earning interest at an annual rate r, compounded M times yearly, then the value of the account after t years is

$$P(t) = P_0\left(1 + \frac{r}{M}\right)^{Mt}$$

The factor $\left(1 + \frac{r}{M}\right)^M$ is called the **yearly multiplier**.

THEOREM 2 Limit Formula for e and e^x

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n \quad \text{and} \quad e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n \quad \text{for all } x$$

Continuously Compounded Interest If P_0 dollars are deposited into an account earning interest at an annual rate r, compounded continuously, then the value of the account after t years is $P(t) = P_0e^{rt}$

The PV of P dollars received at time t is Pe^{-rt}.

PV of an Income Stream If the interest rate is r, the present value of an income stream paying out $R(t)$ dollars per year continuously for T years is

$$\text{PV} = \int_0^T R(t)e^{-rt}\,dt$$

Calculus Notes 5.7 & 5.8: Further Transcendental Functions & Exponential Growth and Decay

Example 5: In the laboratory, the number of *Escherichia coli* bacteria grows exponentially with growth constant $k = 0.41(hours)^{-1}$. Assume that 1000 bacteria are present at time t=0.

a. Find the formula for the number of bacteria P(t) at time t.

b. How large is the population after 5 hours?

c. When will the population reach 10,000?

Calculus Notes 5.7 & 5.8: Further Transcendental Functions & Exponential Growth and Decay

Example 6: Find all solutions of $y' = 3y$. Which solution satisfies $y(0) = 9$?

Example 7: The isotope radon-222 decays exponentially with a half-life of 3.825 days. How long will it take for 80% of the isotope to decay?

Calculus Notes 5.7 & 5.8: Further Transcendental Functions & Exponential Growth and Decay

Example 8: Pharmacologists have shown that penicillin leaves a person's bloodstream at a rate proportional to the amount present.

a. Express this statement as a differential equation.

b. Find the decay constant if 50 mg of penicillin remains in the bloodstream 7 hours after an initial injection of 450 mg.

c. Under the hypothesis (b), at what time was 200 mg of penicillin present?

Calculus Notes 5.7 & 5.8: Further Transcendental Functions & Exponential Growth and Decay

Example 9: A computer virus nicknamed the *Sapphire Worm* spread throughout the internet on January 25, 2003. Studies suggest that during the first few minutes, the population of infected computer hosts increased exponentially with growth constant

$$k = 0.0815s^{-1}$$

a. What was the doubling time for this virus?

b. If the virus began in 4 computers, how many hosts were infected after 2 minutes? 3 minutes?

Example 10: In 1940, a remarkable gallery of prehistoric animal paintings was discovered in the Lascaux cave in Dordogne, France. A charcoal sample from the cave walls had a $C^{14}-to-C^{12}$ ratio equal to 15% of that found in the atmosphere. Approximately how old are the paintings?

Calculus Notes 5.7 & 5.8: Further Transcendental Functions & Exponential Growth and Decay

Example 11: Is it better to receive $2000 today or $2200 in 2 years? Consider $r=0.03$ and $r=0.07$.

Example 12: Chief Operating Officer Ryan Martinez must decide whether to upgrade his company's computer system. The upgrade costs $400,000 and will save $150,000 a year for each of the next 3 years. Is this a good investment if $r=7\%$?