Functions are one of our most important tools for analyzing phenomena. Biologists have studied the antler weight of male red deer as a function of age.
Calculus Notes 4.1: Linear Approximations and Applications.

\[ \Delta f = f(a + \Delta x) - f(a) \]

Linear Approximation of \( \Delta f \)  
If \( f \) is differentiable at \( x = a \) and \( \Delta x \) is small, then

\[ \Delta f \approx f'(a)\Delta x \]

where \( \Delta f = f(a + \Delta x) - f(a) \).

**FIGURE 1** Graphical meaning of the Linear Approximation \( \Delta f \approx f'(a)\Delta x \).
Example 1: Use Linear Approximation to estimate \( \frac{1}{9.3} - \frac{1}{9} \). How accurate is your estimate?

\[
\Delta f \approx f'(a)\Delta x \quad a = 9 \quad \Delta x = 0.3 \quad f(x) = \frac{1}{x} \quad f'(x) = -\frac{1}{x^2}
\]

Linear Approximation:
\[
\Delta f \approx f'(9)(0.3) = -\frac{1}{9^2}(0.3) \approx -0.0037037
\]

Actual Change:
\[
\Delta f = f(9 + 0.3) - f(9) = \frac{1}{9.3} - \frac{1}{9} = -0.003584
\]

Error:
\[
\left| -0.0037037 - (-0.003584) \right| = 0.0001197
\]

Percent Error:
\[
\frac{0.0001197}{-0.003584} \approx 0.033398 \approx 3.33\% 
\]
Calculus Notes 4.1: Linear Approximations and Applications.

**Differential Notation:** The Linear Approximation to \( y = f(x) \) is often written the “differentials” \( dx \) and \( dy \). In this notation \( dx \) is used instead of \( \Delta x \) to represent change in \( x \), and \( dy \) is the corresponding vertical change in the tangent line.

\[
dy = f'(a)dx
\]

Let \( \Delta y = f(a + dx) - f(a) \) Then the Linear approximation says \( \Delta y \approx dy \).

This is simply another way of writing \( \Delta f = f'(a)\Delta x \).
Example 2: The radius of a circular disk is given as 24 cm with a maximum error in measurement of 0.2 cm. Use differentials to estimate the maximum error in the calculated area of the disk.

\[ A = \pi r^2 \]
\[ r = 24 \]
\[ dr = 0.2 \]

\[ dA = 2\pi r (dr) \]
\[ dA = 2\pi (24)(0.2) \]
\[ dA = 9.6\pi \approx 30cm^2 \]

So maximum possible error in the calculated area of the disk is about 30cm².
**Calculus Notes 4.1: Linear Approximations and Applications.**

**Example 3:** A thin metal cable has length $L=12\text{cm}$ when the temperature is $T=21^\circ C$. Estimate the change in length when $T$ rises to $24^\circ C$, assuming that $\frac{dL}{dT} = kL$

Where $k = 1.7 \times 10^{-5}^\circ C^{-1}$ (k is called the coefficient of thermal expansion.)

\[
\frac{dL}{dT} \bigg|_{L=12} = kL = \left(1.7 \times 10^{-5}\right)(12) \approx 0.000204 \text{ cm} / ^\circ C
\]

\[
\approx 2 \times 10^{-4} \text{ cm} / ^\circ C
\]

The Linear Approximation $\Delta L \approx dL$ tells us that the change in length is approximately

\[
\Delta L \approx \frac{dL}{dT} dT \approx \left(2 \times 10^{-4}\right)(3) = 6 \times 10^{-4} \text{ cm}
\]
Calculus Notes 4.1: Linear Approximations and Applications.

**Approximating** \( f(x) \)** by Its Linearization.** If \( f \) is differentiable at \( x = a \) and \( x \) is close to \( a \), then

\[
f(x) \approx L(x) = f'(a)(x-a) + f(a)
\]

Keep in mind that linearization and the Linear Approximation are two ways of saying the same thing. Indeed, when we apply the linearization with \( x = a + \Delta x \) and re-arrange, we obtain the Linear Approximation:

\[
f(x) \approx f(a) + f'(a)(x-a)
\]

\[
f(a + \Delta x) \approx f(a) + f'(a)\Delta x \quad \text{(since } \Delta x = x - a \text{)}
\]

\[
f(a + \Delta x) - f(a) \approx f'(a)\Delta x
\]
The linearization at \( a=1 \) is:
\[
L(x) = f(1) + f'(1)(x - 1)
\]
\[
L(x) = 1 + \left( \frac{3}{2} \right)(x - 1)
\]
The linearization can be used to approximate function values.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \sqrt{xe^{x-1}} )</th>
<th>Linearization at ( a = 1 ): ( L(x) = \frac{3}{2}x - \frac{1}{2} )</th>
<th>Calculator</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>( \sqrt{1.1e^{0.1}} )</td>
<td>( L(1.1) = \frac{3}{2}(1.1) - \frac{1}{2} = 1.15 )</td>
<td>1.15911</td>
<td>( 10^{-2} )</td>
</tr>
<tr>
<td>0.999</td>
<td>( \sqrt{0.999e^{-0.001}} )</td>
<td>( L(0.999) = \frac{3}{2}(0.999) - \frac{1}{2} = 0.9985 )</td>
<td>0.998501</td>
<td>( 10^{-6} )</td>
</tr>
<tr>
<td>2.5</td>
<td>( \sqrt{2.5e^{1.5}} )</td>
<td>( L(2.5) = \frac{3}{2}(2.5) - \frac{1}{2} = 3.25 )</td>
<td>7.086</td>
<td>3.84</td>
</tr>
</tbody>
</table>
Calculus Notes 4.1: Linear Approximations and Applications.

\[ \Delta f \approx f'(a) \Delta x \quad \text{(for } \Delta x \text{ small)} \]

\[ \Delta y \approx dy \quad \text{(for } dx \text{ small)} \]

\[ L(x) = f'(a)(x - a) + f(a) \]

\[ f(x) \approx L(x) \quad \text{(for } x \text{ close to } a) \]

PS 4.1 pg.213 #3, 4, 5, 7, 9, 11, 13, 15, 17, 20, 21, 31, 37 (13)